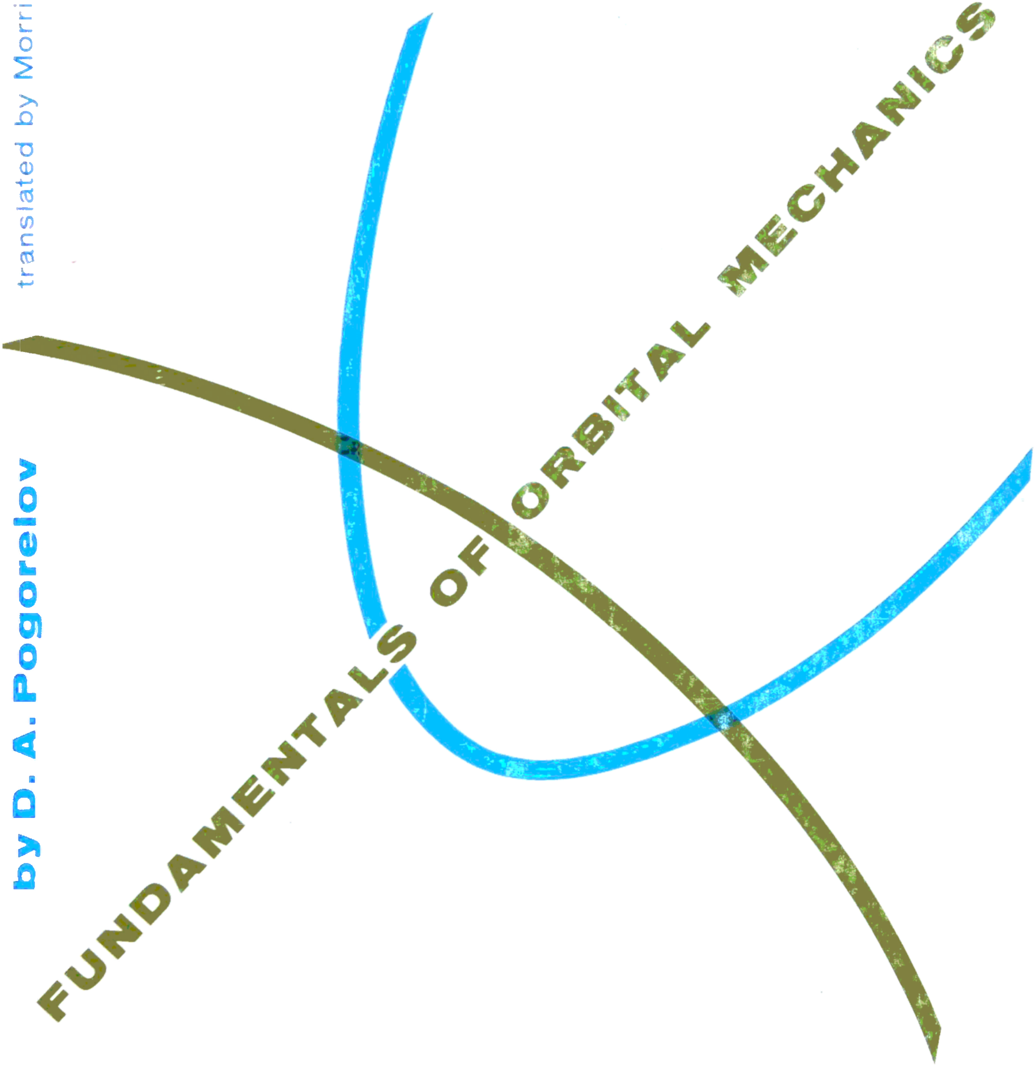
# FUNDAMENTALS OF ORBITAL MECHANICS

by D. A. Pogorelov

translated by Morris Friedman



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# FUNDAMENTALS

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by D. A. Pogorelov

# OF ORBITAL MECHANICS

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# PREFACE

At the present time, when the real possibilities of conquering cosmic space have been disclosed by the world-renowned achievements of Soviet science and engineering, there is great interest in the dynamic problems of near-earth and cosmic flights. It is expedient to start the study of complex space flights with the study of Keplerian motions, the simplest kinds of motions in cosmic space. Among a large number of readers interested in the problems of flight in outer space, the need has been felt for a small book on the theory of Keplerian motions, in which all kinds of Keplerian motions and their characteristics that are of interest in connection with space flights would be considered in a compressed but sufficiently complete form.

The present book is an attempt to satisfy the above-mentioned demand. It should be understandable to all those acquainted with the mathematics courses of the higher technical institutes as well as to students in the senior courses of these institutes.

A small list of the basic literature on astronautics and the classical two-body problem existing in Russian is presented at the end of the book.

In writing the book the author used material from lectures on the mechanics of space flight intended for students of the Lomonosov University of Moscow, with the kind permission of D. E. Okhotsimskii, Corresponding Member of the USSR Academy of Science. Some of the material was used in the explanation of sections 3, 4, 7, 10 and 11. Moreover, comments and remarks of Docent I. I. Razin and Engineer B. F. Zhdaniuk, who attentively read the book in manuscript, contributed to an improvement in the explanation. In this connection, the author takes this opportunity to extend to them his full gratitude.

2 April 1961 D. Pogorelov

# INTRODUCTION

Among the many problems of near-earth and cosmic flights, one problem occupies a very special place: that of studying the free motion of a flying vehicle in such proximity to the celestial body under investigation that all the forces acting on the freely flying vehicle, except the force of attraction produced by the celestial body, can be neglected. The peculiarity of this problem, in contrast to the majority of other problems of near-earth and cosmic flights that present great difficulties to investigators, is that it reduces to the well-studied two-body problem of celestial mechanics and presents no great difficulties as to mathematical description and study. Hence, all the motions included within the scope of the two-body problem are usually called **Keplerian motions** (after J. Kepler, who discovered the fundamental laws of the elliptical motion of the planets, which are also valid for a wider class of motions). Thus, Keplerian motions are the simplest kinds of motions of flying vehicles in cosmic space. It is known that the trajectories of the more complex space flights can be represented, in a first approximation, as consisting of Keplerian trajectories. Thus, for example, the flight trajectory to any of the planets of the solar system can be represented as consisting of an initial Keplerian orbit with a focus at the center of the earth, an intermediate Keplerian orbit with a focus at the center of the sun, and a final Keplerian orbit with focus at the center of the planet to be reached.

It is hence natural that the theory of space flights should start with the theory of Keplerian motions.

In the scheme of Keplerian motions, the flying vehicle is taken as a point mass moving in a plane passing through the center of the celestial body producing an attractive force. The study of the properties of Keplerian motions and the establishment of fundamental dependences between their characteristics when analyzing the motion of flying vehicles in this plane is the basic content of the present book. We start the explanation with the formulation of the requisite theorems of the mechanics of a point mass, the determination of the attractive force of the celestial body which has a spherical mass distribution configuration, the formulation of the classical two-body problem, and the derivation of the trajectory equation for Keplerian motions. Then a derivation will be given of the relations connecting the trajectory parameters to the initial motion conditions and the magnitude and direction of the flight velocity; the angular range and the flight time for any current point of motion are determined. Furthermore, we analyze the family of trajectories of identical initial velocity and identical range and we determine the maximum range and the minimum initial velocity trajectory parameters. Finally, at the end of the book relations will be derived which connect angular range and flight time deviations with initial motion deviations.

We shall take the flying vehicle to be a free point mass. It will hence be convenient to start the explanation not with an analysis and integration of the equations obtained by projecting the original vector equation describing the point motion onto the axes of some coordinate system, but with an application of fundamental theorems of mechanics of a free point mass to the kind of motion to be studied.

# 1. FUNDAMENTAL THEOREMS ON THE MECHANICS OF A FREE POINT MASS

The direct integration of the equations of motion of a free point mass, which are obtained by projecting the initial vector equation

$$(1.1)\qquad m\frac{d^2\vec{r}}{dt^2} = \vec{F}$$

onto the axes of some coordinate system, where m is the mass of the point, $\vec{r}$ the radius vector of the point, $\vec{F}$ the total force acting on the point, and t the time, is often very difficult. In certain particular, but important, cases these difficulties can be eliminated by using a transformation of the initial vector equation (1.1). The general dependences obtained by the transformation of the initial equations of motion of the point, and which permit the first integrals of the initial equation to be obtained in certain particular, but often encountered, kinds of forces are called the general theorems of the mechanics of a point mass.

The problem under consideration below refers directly to such a case, when it is convenient to use theorems on the kinetic moment and on the kinetic energy.

1. Kinetic Moment Theorem or the Theorem of Areas. This theorem is obtained by the following simple transformation of the original equation (1.1). By multiplying both sides vectorially by $\vec{r}$ , we obtain

$$\vec{r} \times m\frac{d^2\vec{r}}{dt^2} = \vec{r} \times \vec{F} ,$$

after which, by using the identity

$$\frac{d}{dt}\left(\vec{r} \times m\frac{d\vec{r}}{dt}\right) = \left(\frac{d\vec{r}}{dt} \times m\frac{d\vec{r}}{dt}\right) + \left(\vec{r} \times m\frac{d^2\vec{r}}{dt^2}\right) = \vec{r} \times m\frac{d^2\vec{r}}{dt^2}$$

we write the desired dependence in the final form

$$\frac{d}{dt}\left(\vec{r} \times m\frac{d\vec{r}}{dt}\right) = \vec{r} \times \vec{F} . \tag{1.2}$$

This dependence is called the **kinetic moment theorem** and is formulated as follows: **The time derivative of the kinetic moment of a point mass relative to a certain center equals the moment, with respect to the same center, of a force under the action of which the point moves.**

This theorem is often given another form also. It is known that the quantity

$$d\vec{\Sigma} = \vec{r} \times d\vec{r}$$

is the oriented, doubled area of a triangle formed by the vectors $\vec{r}$, $d\vec{r}$ and $\vec{r} + d\vec{r}$, or equivalently, the oriented, doubled area of a sector described by the radius vector $\vec{r}$ in the time dt (Fig. 1). The derivative

$$\vec{\sigma} = \frac{d\vec{\Sigma}}{dt} = \vec{r} \times \frac{d\vec{r}}{dt} \tag{1.3}$$

is called the **doubled sectorial velocity**. By expressing the left side of (1.2) in terms of the doubled sectorial velocity, we obtain

$$m\frac{d\vec{\sigma}}{dt} = \vec{r} \times \vec{F} . \tag{1.4}$$

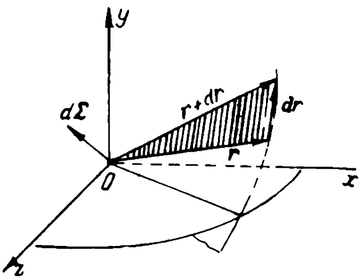

Fig. 1

Since the sectorial velocity expresses the velocity at which a certain area is swept by a radius vector, the kinetic moment theorem is called the **theorem of areas** in later treatments. It can be formulated in the following manner: **The product of the mass of a material point by its doubled sectorial acceleration, defined with respect to a certain center, equals the moment, with respect to the same center, of a force under whose action the point moves.**

If the moment of the force acting on the point mass equals zero, with respect to a certain center, during the whole time the point moves, then the kinetic moment with respect to this center remains constant on the basis of (1.2):

$$(1.5) \qquad \vec{r} \times m\frac{d\vec{r}}{dt} = c\,.$$

A force with this property is called a **central force**; the center relative to which the moment equals zero is the **center of the force**; the motion performed by the point under the action of the central force is **central motion.**

Hence, in the case of a central force the kinetic moment theorem yields one vector or three scalar first integrals of the differential equations of motion of a point mass.

According to (1.3), the vector integral (1.5) can also be represented in another form:

$$(1.6) \qquad \vec{\sigma} = \frac{\vec{c}}{m}\,,$$

which shows the constancy of the doubled sectorial velocity $\vec{\sigma}$ in the case of a central force. The vector integral (1.6) and the three corresponding scalar integrals are called **area integrals.**

2. Theorem of the Kinetic Energy or Vis Viva Theorem. This theorem is also obtained simply by using a transformation of the original equation (1.1).

Let us write the obvious equality

$$(1.7) \qquad \frac{d\vec{r}}{dt}\,dt = d\vec{r}\,.$$

Multiplying the left and right sides of (1.1) and (1.7), we obtain:

$$m(\vec{v}\cdot d\vec{v}) = \vec{F}\cdot d\vec{r}$$

where $\vec{v} = \dfrac{d\vec{r}}{dt}$ .

Finally, let us write the desired dependence thus:

$$(1.8) \qquad d\left(\frac{mv^2}{2}\right) = \vec{F} \cdot d\vec{r} \ .$$

This dependence is called the **kinetic energy theorem** or the **vis viva theorem**. It is formulated as follows: **The differential of the kinetic energy of a point mass equals the element of work by a force acting on a point.**

If the force acting on the point mass has a force function U and, therefore, the element of work can be expressed by the total differential of the force function

$$\vec{F} \cdot d\vec{r} = dU \ ,$$

then (1.8) can be integrated directly and leads to the first integral of the equation of motion of a point

$$\frac{mv^2}{2} = U + h$$

where h is an arbitrary constant. This integral is called the **energy** or the **vis viva integral.**

## 2. ATTRACTIVE FORCE OF A MATERIAL BODY HAVING SPHERICAL STRUCTURE. BASIC EARTH ATTRACTIVE FORCE

According to the law of universal gravitation, also known as Newton's law, each two material particles are mutually attracted with a force proportional to their masses and inversely proportional to the square of the distance between them.

Let M and m be the masses of two material particles, and r their mutual distance; then each particle is acted upon by a force directed toward the other particle which equals

(2.1) $$F = f\frac{Mm}{r^2}$$

where f is the **universal gravitational constant.**

Often only the attractive force acting on one particle, the particle with mass m, say, must be examined. The effect of the particle of mass M on the particle of mass m causes this force, and when m = 1

(2.2) $$F = \frac{fM}{r^2}$$

It is known from mechanics that the previously mentioned force is conservative, i.e., has a force function. The force function of the force (2.2) is called the **Newtonian potential** of the particle of mass M and it equals

(2.3) $$\vec{V} = f\frac{M}{\vec{r}}$$

where $\vec{r}$ should be considered a spherical coordinate of the particle of mass m = 1 in a coordinate system in which the particle of mass M is taken as the origin.*

It follows from the properties of the force function that the projection of the force (2.2) on an arbitrary x-axis is

(2.4) $$F_x = \frac{\partial V}{\partial x} .$$

In particular, the projection of the force (2.2) on the $\vec{r}$-axis directed from the particle with mass M toward the particle with unit mass equals

(2.5) $$F_r = \frac{\partial V}{\partial r} .$$

The physical meaning of the potential (2.3) is easily expounded by using (2.5).

---

*From the viewpoint of the terminology used in theoretical mechanics, this function should be called the Newtonian work function. However, the somewhat inaccurate designation "potential" has been inculcated in the extensive literature on the theory of the Newtonian attraction, and we shall also retain it.

Let us consider a force F' which is equal in magnitude to the force F with which the particle of mass M attracts the particle of unit mass but is opposite in direction. Evidently

$$(2.6) \qquad F'_r = -\frac{\partial V}{\partial r} .$$

Let us apply the force F' to a particle of unit mass and let us determine the work performed in displacing this particle from a certain initial position, characterized by the quantity r, to infinity. As is shown in courses on theoretical mechanics, the work performed by a conservative force is independent of the path and is determined completely by the positions of the initial and final points. Hence, the desired work will be independent of the trajectory of the displaced particle because of the conservative nature of the force F' and, for simplicity of the calculations, the particle can be removed to infinity along a straight line passing through the particle of unit mass and the particle of mass M. In this case the work to be determined is

$$(2.7) \qquad A = \int_r^{\infty} F' \, dr = V(r) - V(\infty) = V(r) .$$

From (2.7) it follows that the Newtonian potential at some point equals the work which must be expended to overcome the force of Newtonian attraction when removing the attracted particle of unit mass from that point to infinity.

Just as the Newtonian potential of a single particle has been introduced above, it is also possible to introduce the Newtonian potential of n particles:

$$(2.8) \qquad V = f \sum_{i=1}^{n} \frac{M_i}{r_i}$$

where $M_i$ is the mass of the i-th particle, $r_i$ the distance of the i-th particle from the particle of unit mass.

Correspondingly, the Newtonian potential of an arbitrary body of mass M can be expressed in the following general form:

$$(2.9) \qquad V = f \int_M \frac{dm}{r}$$

where r is the distance of the body particles which have mass dm from the particle of unit mass, and the integration is taken over the entire mass of the body.

The problem of determining the integral (2.9) for a body of complex shape and complex mass distribution is very difficult. However, it is easily solved for bodies having a spherical structure.

A body which has the shape of a sphere and for which the density at all points equidistant from the center is identical will be called a **body of spherical structure.**

Let R denote the distance of an arbitrary point of a body of spherical structure from its center and $\rho$ the density at this point. Then, evidently

$$\rho = \rho(R)$$

Let us describe a spherical surface of radius R from the center of the body A (Fig. 2). Furthermore, taking as axis the line AO connecting the center A to the particle being attracted, O, let us describe two conical surfaces ABB' and ACC' with the common vertex A and vertex angles $2\varphi$ and $2(\varphi + d\varphi)$, respectively.

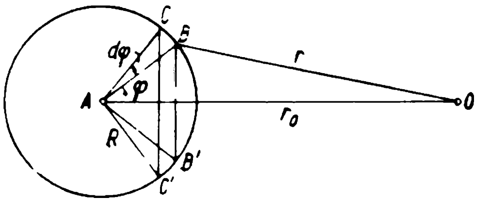

Fig. 2

The surface area of the spherical band BCC'B' cut by these cones will equal

$$2\pi R \sin \varphi \, R \, d\varphi \, .$$

Furthermore, let us describe an analogous spherical surface of radius R + dR. The mass of the part of the body enclosed between these two spheres and two cones will evidently equal

$$dm = \rho \, 2\pi R^2 \sin \varphi \, dR \, d\varphi \, .$$

The distance r of all the particles of the part of the body thus isolated from the attracted particle O can be considered identical. Then

$$V = f \int_M \frac{dm}{r} = 2\pi f \int_0^{R_0} \rho(R)\ R^2 dR \int_0^{\pi} \frac{\sin\ \varphi\ d\varphi}{r}$$

where $R_0$ is the radius of the whole body under consideration.

Setting the distance AO equal to $r_0$, let us write

$$r^2 = R^2 + r_0^2 - 2Rr_0 \cos\ \varphi\ .$$

After this, by replacing the variable of integration $\varphi$ by the variable r which will vary between the limits $r_0 - R$ to $r_0 + R$, we obtain:

$$r\ dr = Rr_0 \sin\ \varphi\ d\varphi$$

and

$$V = \frac{2\pi f}{r_0} \int_0^{R_0} \rho(R) R dR \int_{r_0-R}^{r_0+R} dr = \frac{4\pi f}{r_0} \int_0^{R_0} \rho(R) R^2 dR$$

Since the mass of a body of spherical structure is

$$M = 4\pi \int_0^{R_0} \rho(R) R^2 dR\ ,$$

then

$$V = f \frac{M}{r_0}\ . \tag{2.10}$$

The expression (2.10) shows that a body of spherical structure attracts an external point just as if all its mass were concentrated at the center.

Since the earth, like other celestial bodies, has a very complex structure which has been studied only slightly, it must be replaced by some simpler model in many cases. In the simplest model used, the earth is represented as a body of spherical structure. In this case the Newtonian potential and the earth's attractive force can be represented as follows:

$$V_0 = \frac{\pi_0}{r} \tag{2.11}$$

$$(2.12) \quad \vec{F}_0 = - m \frac{\pi_0}{r^3} \vec{r}$$

where m is the mass of the particle being attracted; $\vec{r}$ the radius vector of the particle drawn from the center of the earth; $\pi_0$ = fM the product of the gravitational constant f by the mass of the earth M. The numerical value of this constant can be taken as $\pi_0 = 3.986 \cdot 10^{14}$ $m^3/sec$ according to the data of many works.

3.98[illegible]

for cm and gram

For brevity, we shall designate the potential and the force defined above as the **fundamental potential** and the **fundamental force of earth's attraction.**

The acceleration communicated to the particle being attracted by the fundamental force of earth's attraction is

$$(2.13) \quad \vec{b}_0 = - \frac{\pi_0}{r^3} \vec{r}$$

We shall call this the **acceleration of the fundamental force of earth's attraction.**

# 3. CLASSICAL TWO-BODY PROBLEM. EQUATION OF THE TRAJECTORY OF A POINT MOVING UNDER THE EFFECT OF THE FUNDAMENTAL FORCE OF EARTH'S ATTRACTION

Let us assume that there are two bodies which are mutually attracted according to Newton's law. Let us also assume that they have a spherical structure. Then on the basis of the exposition in the preceding section, the attraction between these bodies can be considered as coincident with the attraction of two points having the masses of these bodies and located at the centers of mass of these bodies. Therefore, the problem of studying the motion of two bodies under the effect of their mutual attraction, which is called the **two-body problem** in mechanics, reduces, for bodies having spherical structure, to the considerably simpler problem of studying the motion of two points under the effect of their mutual attraction. The simple problem involving spherical bodies is usually called the classical two-body problem, to distinguish it from the general two-body problem. In those works where only this problem is analyzed and, therefore, there is no necessity to distinguish it from more general problems, it is usually called simply the two-body problem. As is well known, it was first solved by I. Newton by a geometric method. D. Bernoulli gave its analytical solution, and finally, it was investigated in detail by L. Euler. Up to now, changes in its solution have involved mainly the choice of variables in terms of which it will be expressed. A detailed explanation of this problem is usually given in all courses on celestial mechanics and theoretical astronomy.

The motion of a projectile or any apparatus launched from the earth's surface into a vacuum space can, in many cases, be reduced to the scheme included within the scope of the classical two-body problem. In particular, the problem of studying the motion of a projectile or any apparatus in a vacuum space near the earth can be represented as a classical two-body problem in which one body has a spherical structure and mass M, equal to the mass of the earth, and the other can be considered as a material point having mass m equal to the mass of the projectile or apparatus. Let us examine this particular case of the classical two-body problem for which mass M is so much larger than mass m that it is possible to neglect the effect of the mass m on the mass M.

Let A denote the center of mass of the earth and S that of the point mass whose motion we shall analyze. Because of the foregoing, the motion of the point S can be considered as taking place under the effect only of the fundamental force of the earth's attraction $F_0$, directed toward the point A which will be considered fixed. The usual means could be used to write the equation of motion of the point S; we take as the original equation

$$m \frac{d^2\vec{r}}{dt^2} = \vec{F}_0$$

and by successive transformation and projection on the appropriate axes we obtain all the dependences of interest. However, this problem can be solved more briefly and simply by using the theorems of the kinetic moment and the kinetic energy.

Thus, by using the kinetic moment theorem it is possible to obtain a vector integral of the equation of motion of the point S in the form of (1.5)

$$\vec{r} \times m\frac{d\vec{r}}{dt} = \vec{c}$$

where $\vec{r}$ is the radius vector of the point S drawn from A and $\vec{c}$ is a certain constant vector.

This integral shows that the point S can perform motion only in a plane passing through the point A and perpendicular to the constant vector $\vec{c}$. It is evident that because of this motion it is convenient to examine the point S in an Axyz coordinate system whose origin is at the center of the earth, the Axy plane being superposed on the plane in which the motion of the point S occurs. In the coordinate system thus selected, the study of the motion of the point S under the effect of the fundamental force of earth's attraction reduces to the study of its motion in the Axy plane. The vector integral written above thereby yields the simplest solution of a second-order differential equation determining the change in the coordinate z of the point under consideration in the form

$$\frac{dz(t)}{dt} = 0; \quad z(t) = 0 .$$

Moreover, since the vector integral is equivalent to three scalars, then as will be shown below, it will also yield a first integral of the equation of motion of the point S in the Axy plane.

Because of the spherical symmetry of the field of fundamental force of the earth's gravitation, it is convenient to examine the motion of the point S in the Axy plane in the polar coordinates r and $\eta$, related to the Ax- and Ay-axes as shown in Fig. 3. In the figure, S is the current position of the point under consideration and $O_1$ is a certain initial position. In our case it is convenient to determine the velocity vector $\vec{v}$ of the moving point S by using the absolute value of this vector v and the angle of its inclination $\theta$ to a plane perpendicular to the radius vector of this point. It is customary here to take the upward direction from the plane perpendicular to the radius vector as the positive direction of measuring $\theta$ and the downward direction (toward the center of the earth) as the negative direction. The angle $\theta$ thus defined is called the angle of deviation. In conformity with the notation introduced, the parameters defining the position and velocity of the point S will be denoted thus: for the current instant, r, $\eta$, v, $\theta$, and for the selected initial instant $r_1$, $\eta_1$, $v_1$, $\theta_1$ .

We must obtain the scalar integral of the equations describing the motion of the point under consideration in the Axy plane from the vector integral written above. Hence, we decompose the radius vector of the point S and its velocity vector into components along the polar coordinate axes:

$$(3.1) \qquad \vec{v} = v_r \vec{r}^0 + v_\eta \vec{\eta}^0 \; ; \qquad \vec{r} = r\vec{r}^0$$

where $\vec{r}^0$ and $\vec{\eta}^0$ are unit vectors of the polar coordinate axes drawn through the point S (Fig. 3).

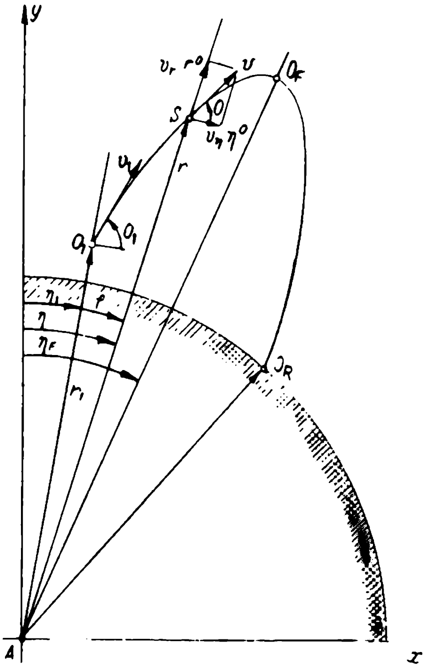

Fig. 3

Using (3.1), let us transform the vector integral mentioned; we obtain:

$$\vec{r} \times m\frac{d\vec{r}}{dt} = m\left[r\vec{r}^0 \times (v_r \vec{r}^0 + v_\eta \vec{\eta}^0)\right]$$

$$= mrv_\eta \,(\vec{r}^0 \times \vec{\eta}^0) = \vec{c} \; .$$

Taking into account that

$$(3.2)\qquad v_r = \frac{dr}{dt}; \qquad v_\eta = r\frac{d\eta}{dt}$$

and that the vector product of the unit vectors $\vec{r}^0$ and $\vec{\eta}^0$ is a unit vector parallel to the constant vector $\vec{c}$ and

$$\frac{\vec{c}}{m} = \vec{\sigma} = \text{const},$$

the desired scalar integral can be represented on the basis of (1.6) in the form

$$(3.3)\qquad r^2\frac{d\eta}{dt} = \sigma = \text{const}.$$

Equation (3.3) shows that the magnitude of the doubled sectorial velocity $\sigma$ of the point S with respect to the center of the earth remains constant throughout its motion.

However, knowledge of the integral (3.3) is still insufficient to determine the trajectory of the point S. The trajectory equation can be written by using, besides this integral, the kinetic energy theorem in the form (1.8):

$$(3.4)\qquad \frac{1}{2}\,d(v^2) = \frac{\vec{F}}{m}\,d\vec{r}\,.$$

In the case we consider, on the basis of (2.12),

$$(3.5)\qquad \frac{\vec{F}}{m} = \frac{\vec{F}_0}{m} = -\frac{\pi_0}{r^2}\,\vec{r}^0\,.$$

Substituting (3.5) into (3.4), we rewrite it thus:

$$\frac{1}{2}\,\frac{d(v^2)}{dt} = -\frac{\pi_0}{r^2}\,\vec{r}^0\,\frac{d\vec{r}}{dt}\,.$$

Using (3.1) for the velocity vector and relations (3.2), let us express the equality of interest in terms of scalar quantities only:

(3.6) $$\frac{1}{2}\frac{d(v^2)}{dt} = -\frac{\pi_0}{r^2}\frac{dr}{dt} .$$

This equality relates the three variables r, v and t. In order to write the trajectory equation in the polar coordinates r and $\eta$, it is necessary to obtain an equation relating only these two variables. Such an equation can be obtained from (3.6) if a transformation is made from the variable t to the variable $\eta$ and then v is expressed in terms of r and $\eta$.

Let us first transform from the independent variable t to the independent variable $\eta$ in (3.6). To do this, we rewrite it as follows:

$$\frac{1}{2}\frac{d(v^2)}{d\eta}\frac{d\eta}{dt} = -\frac{\pi_0}{r^2}\frac{dr}{d\eta}\frac{d\eta}{dt} .$$

This equality is equivalent to the two equalities:

(3.7) $$\frac{d\eta}{dt} = 0$$

and

(3.8) $$\frac{1}{2}\frac{d(v^2)}{d\eta} = -\frac{\pi_0}{r^2}\frac{dr}{d\eta} .$$

The first equality permits us to obtain the trajectory equation directly for the simplest particular case of point motion, the case of rectilinear motion

(3.9) $$\eta = \eta_0 = \text{const} .$$

The second equality permits the elimination of the variable t and, later, of the variable v and permits us to obtain the trajectory equation for the general case of point motion.

In order to express the quantity v in terms of $\eta$ and r, we use the expression

(3.10) $$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\eta}{dt}\right)^2 ,$$

which can easily be obtained from (3.1) and (3.2). Representing (3.10) in the form

$$(3.11)\qquad v^2 = \left(\frac{d\eta}{dt}\right)^2 \left[\left(\frac{dr}{d\eta}\right)^2 + r^2\right]$$

and using the integral (3.3), we eliminate the variable t:

$$(3.12)\qquad v^2 = \frac{\sigma^2}{r^4} \left[\left(\frac{dr}{d\eta}\right)^2 + r^2\right]$$

Then, substituting (3.12) into (3.8), we obtain the desired trajectory equation. However, before this it is convenient to replace r by the new variable

$$(3.13)\qquad u = \frac{1}{r}$$

which can be used to write the differential equation of the trajectory in its simplest form. Transforming to the variable u in (3.8) and (3.12) by using the expressions

$$(3.14)\qquad r = \frac{1}{u} \quad \text{and} \quad \frac{dr}{d\eta} = -\frac{1}{u^2}\frac{du}{d\eta} ,$$

we obtain

$$(3.15)\qquad \frac{1}{2}\frac{d(v^2)}{d\eta} = \frac{du}{d\eta}\pi_0 ; \quad v^2 = \sigma^2 \left[\left(\frac{du}{d\eta}\right)^2 + u^2\right]$$

Now substituting the expression for $v^2$ in the first equality, the desired equation can be written thus:

$$(3.16)\qquad \left(\frac{d^2u}{d\eta^2} + u\right)\frac{du}{d\eta} = \frac{\pi_0}{\sigma^2}\frac{du}{d\eta} .$$

The equality (3.16) permits the trajectory equation to be obtained for the particular case of point motion along a circle

$$(3.17)\qquad \frac{du}{d\eta} = 0 \quad \text{or} \quad \frac{dr}{d\eta} = 0$$

or

(3.18) $r = r_0 = \text{const}$ .

For the general case of point motion, on the basis of (3.16) we can represent the desired differential equation of the trajectory as follows:

$$\frac{d^2u}{d\eta^2} + u = U_0 \tag{3.19}$$

where

$$U_0 = \frac{\pi_0}{\sigma^2} = \text{const} . \tag{3.20}$$

The general solution of the linear differential equation (3.19) with constant coefficients can be written as:*

$$u = U_0 - U_1 \cos\eta - U_2 \sin\eta \tag{3.21}$$

where $U_1$ and $U_2$ are integration constants and $U_0$ is the particular solution of the inhomogeneous equation (3.19); or it can be reduced to the form:

$$u = U_0 \left[1 - e \cos(\eta - \eta_F)\right] \tag{3.22}$$

where the constants e and $\eta_F$ can be expressed in terms of the constants $U_0$, $U_1$, $U_2$ as follows:

$$e^2 = \frac{U_1^2 + U_2^2}{U_0^2} ; \tag{3.23}$$

$$\cos\eta_F = \frac{U_1}{\sqrt{U_1^2 + U_2^2}} ; \quad \sin\eta_F = \frac{U_2}{\sqrt{U_1^2 + U_2^2}} \tag{3.24}$$

(The geometric meaning of $\eta_F$ will be clarified below.)

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*For example, see V. I. Smirnov: Kurs vysshei matematiki (Course in Higher Mathematics) vol. 2, Gostekhizdat, 1951, sections 27, 28.

The relations (3.23) and (3.24) can easily be obtained by comparing the expressions (3.21) and (3.22).

Transforming from the auxiliary variable u to the variable r in (3.22) and introducing the new constant

$$(3.25) \quad p = \frac{1}{U_0}$$

let us write the trajectory equation in the final form

$$(3.26) \quad r = \frac{p}{1 - e \cos (\eta - \eta_F)} .$$

Comparing the obtained equation with the equation

$$(3.27) \quad r = \frac{p}{1 - e \cos \beta}$$

which is the equation of a conic section in polar coordinates,* we arrive at the conclusion that the **trajectory of a point moving under the effect of the fundamental force of earth's attraction is a conic section.**

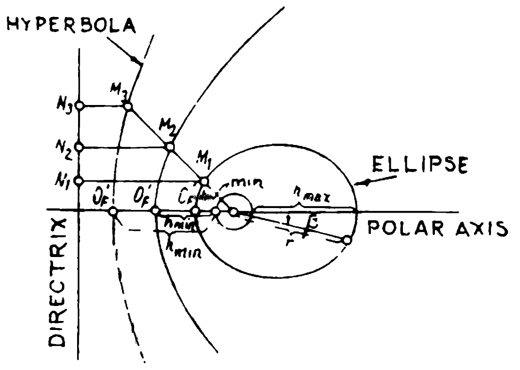

Fig. 4

Let us recall that in writing (3.27), one of the foci of the conic section is taken as the pole of the polar coordinate system and the focal axis of the conic section is taken as the polar axis. The line from the focus chosen as the pole drawn through the other focus is chosen as the positive direction of the polar axis for the ellipse, and the direction from the focus chosen as the pole to infinity on the branch of the curve under consideration is taken for the parabola and hyperbola (Fig. 4).

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*For example, see I. I. Privalov: Analiticheskaya geometriya (Analytic Geometry), Gostekhizdat, 1945, Chapter 4, section 4.

Let us clarify the geometric meaning of the parameters in the trajectory equation. As is known, a conic section can be defined as the geometric locus of points, the ratio of whose distances from a given point (focus) to the distance from a given line (directrix) is a constant value, equal to e. Hence (see Fig. 4):

$$\text{for the ellipse} \quad e = \frac{FM_1}{M_1N_1} < 1 \; ;$$

$$\text{for the parabola} \quad e = \frac{FM_2}{M_2N_2} = 1 \; ;$$

$$\text{for the hyperbola} \quad e = \frac{FM_3}{M_3N_3} > 1 \; .$$

Therefore, the parameter e determines the shape of the curve. It is called the **eccentricity of the conic section.**

As is seen from (3.27), the quantity p equals the length of the radius vector of a curve drawn from the focus and perpendicular to the focal axis. Hence, the parameter p defines the linear dimension of the curve. It is called the **focal parameter of a conic section.**

As is seen from (3.26) and (3.27), the quantity $\eta_F$ equals the angle formed by the focal axis of the conic section and the polar axis of the coordinate system selected for the writing of equation (3.26). Hence, the parameter $\eta_F$ defines the position of the focal axis of the conic section relative to the selected polar axis.

Summarizing, we can make the following deduction: **The trajectories of points moving under the effect of the fundamental force of the earth's attraction can, depending on the initial conditions, be radial, circular, elliptical, hyperbolic or parabolic.** For brevity, all these trajectories are called **Keplerian trajectories** and the motions performed by the point mass along these trajectories are **Keplerian motions.**

# 4. RELATION OF THE FUNDAMENTAL PARAMETERS OF KEPLERIAN TRAJECTORIES TO THE INITIAL MOTION CONDITIONS

In order to determine the influence of the initial conditions of the point motion on the shape, magnitude and location of the trajectory, it is sufficient to write the relations connecting the fundamental trajectory parameters e, p and $\eta_F$ with the point-motion parameters corresponding to a certain instant taken as the initial instant. Since any arbitrary instant of motion can be selected as the initial instant, we will not make it definite here but will write the desired relations in a form valid for any time of the point motion under examination.

Let us introduce the nondimensional point-motion parameter

$$(4.1) \qquad k = \frac{\frac{v^2}{2}}{\frac{\pi_0}{r}} = \frac{v^2 r}{2\pi_0}$$

which is the ratio of the kinetic energy of the point at any instant of the motion to the work which must be expended to transport this point to infinity from the position it occupies at the instant under consideration. This physical interpretation of the parameter k follows from (2.11) for the earth's Newtonian potential and from (2.7) which is valid for any Newtonian potential. The convenience of taking this parameter into consideration is explained by the fact that the trajectory shape is completely defined by its magnitude alone. We arrive at this conclusion somewhat later, but it can naturally be expected by starting from the physical meaning of the parameter.

The motion under consideration can be defined completely if the coordinates of the moving point and its velocity vector are assigned at an arbitrary instant, say, by using the quantities

$$(4.2) \qquad r, \ \eta, \ v, \ \theta$$

introduced in the preceding section (Fig. 3).

By expressing the trajectory parameters e, p and $\eta_F$ in terms of the quantities in (4.2), we obtain the desired relations. To solve the last problem, let us first express the integration constants

$$U_0, U_1, U_2 \tag{4.3}$$

in terms of the quantities in (4.2). Returning to (3.21), we write

$$u = U_0 - U_1\cos\eta - U_2\sin\eta \; ; \; \frac{du}{d\eta} = U_1\sin\eta - U_2\cos\eta \; . \tag{4.4}$$

Moreover, it follows from (3.20) that

$$U_0 = \frac{\pi_0}{\sigma^2} \; . \tag{4.5}$$

Making use of (3.3), (3.13) and (3.14), we write:

$$\begin{cases} \sigma = r^2 \dfrac{d\eta}{dt} \; ; \quad u = \dfrac{1}{r} \\ \\ \dfrac{du}{d\eta} = - u^2 \dfrac{dr}{d\eta} = - \dfrac{1}{r^2} \dfrac{\dfrac{dr}{dt}}{\dfrac{d\eta}{dt}} \; . \end{cases} \tag{4.6}$$

It follows from Fig. 3 that

$$v_r = v \sin\theta \; ; \; v_\eta = v \cos\theta \tag{4.7}$$

and from (3.2) that

$$\frac{dr}{dt} = v \sin\theta \; ; \; \frac{d\eta}{dt} = \frac{v\cos\theta}{r} \; . \tag{4.8}$$

Substituting (4.8) into (4.6), we obtain:

$$\begin{cases} \sigma = rv\cos\theta \\ \\ u = \dfrac{1}{r} \; ; \; \dfrac{du}{d\eta} = - \dfrac{\tan\theta}{r} \end{cases} \tag{4.9}$$

Finally, substituting (4.9) into (4.5) and (4.4), let us write the following relations to determine the constants of (4.3):

$$(4.10)\quad \left\{\begin{aligned} U_0 &= \frac{\pi_0}{\sigma^2} = \frac{\pi_0}{r^2v^2\cos^2\theta} \\ U_1\cos\eta + U_2\sin\eta &= U_0 - \frac{1}{r} \\ U_1\sin\eta - U_2\cos\eta &= -\frac{\tan\theta}{r} \end{aligned}\right.$$

from which, by taking (4.1) into account, we obtain

$$(4.11)\quad \begin{aligned} U_0 &= \frac{\pi_0}{\sigma^2} = \frac{\pi_0}{r^2v^2\cos^2\theta} = \frac{1}{2kr\cos^2\theta} \\ U_1 &= \left(U_0 - \frac{1}{r}\right)\cos\eta - \frac{\tan\theta}{r}\sin\eta \\ U_2 &= \left(U_0 - \frac{1}{r}\right)\sin\eta + \frac{\tan\theta}{r}\cos\eta \\ U_1^2 + U_2^2 &= \left(U_0 - \frac{1}{r}\right)^2 + \frac{\tan^2\theta}{r} . \end{aligned}$$

Using (3.25), (3.23), (3.24) and (4.1) after this, the desired relations can be written in the following general form:

$$(4.12)\quad p = \frac{1}{U_0} = \frac{\sigma^2}{\pi_0} = 2kr\cos^2\theta$$

$$(4.13)\quad \left\{\begin{aligned} e^2 &= \frac{U_1^2 + U_2^2}{U_0^2} = \left(1 - \frac{1}{rU_0}\right)^2 + \left(\frac{\tan\theta}{rU_0}\right)^2 ; \\ e^2 &= (1 - 2k\cos^2\theta)^2 + (2k\sin\theta\cos\theta)^2 \\ &= (1 - 2k\cos^2\theta)^2 + 4k^2(1 - \cos^2\theta)\cos^2\theta \\ &= 1 - 4k\cos^2\theta + 4k^2\cos^2\theta \end{aligned}\right.$$

from which

$$(4.14)\quad e^2 = 1 - 4k(1 - k)\cos^2\theta$$

or

$$(4.15) \quad e^2 = (1 - 2k)^2 \cos^2\theta + \sin^2\theta$$

$$(4.16) \quad \tan \eta_F = \frac{U_2}{U_1} = \frac{(U_0 r - 1)\sin\eta + \tan\theta \cos\eta}{(U_0 r - 1)\cos\eta - \tan\theta \sin\eta} .$$

After having written the desired expressions, it is possible to write the dependence of the shape, magnitude and location of the trajectory on the initial conditions of the motion of the point under consideration.

Let us determine the condition guaranteeing point-motion along the rectilinear trajectory for which we obtained (3.7) or (3.9) in the previous section. It follows from (3.7) and (4.8) that motion along a rectilinear trajectory is possible only when there is compliance with the equality

$$(4.17) \quad \theta = \frac{\pi}{2}$$

even at any initial instant of the motion.

Now, let us determine the condition guaranteeing point-motion along a circle. The trajectory equation for this case was written in the preceding section in the form of (3.17) or (3.18). It follows from (3.17) and (4.9) that motion along a circular trajectory is possible only when the equality

$$(4.18) \quad \theta = 0$$

is always satisfied, even at any initial instant. However, compliance with condition (4.18) at some initial instant, which is the necessary condition, is not sufficient for circular motion, since it does not guarantee zero eccentricity. The additional condition, which must be satisfied at the selected initial instant for the trajectory to be circular, is also easily obtained from (4.15), with (4.18) taken into account:

$$e^2 = (1 - 2k)^2 \cos^2\theta + \sin^2\theta = (1 - 2k)^2 = 0,$$

from which

$$(4.19) \quad k = \frac{rv^2}{2\pi_0} = \frac{1}{2} .$$

Evidently this condition must be satisfied at any instant for motion along a circle, since any instant can be taken as the initial instant of the motion.

In all the remaining cases, the trajectory shape is determined completely by the magnitude of the parameter k and, as is easily seen, from (4.15):

Always $e < 1$ for $0 < k < 1$, i.e., the trajectory is an ellipse;

Always $e = 1$ for $k = 1$, i.e., the trajectory is a parabola;

Always $e > 1$ for $k > 1$, i.e., the trajectory is a hyperbola.

This result corresponds completely to the physical deductions which can be made by starting only from the physical meaning of the parameter k.

In conclusion, let us recall that a very convenient parameter for elliptical trajectories is also the semi-major axis a which is expressed as follows in terms of the parameters p and e:

$$(4.20) \qquad a = \frac{p}{1 - e^2} .$$

Substituting (4.12) and (4.13) into (4.20), we obtain

$$(4.21) \qquad a = \frac{r}{2(1 - k)} .$$

If it is agreed to consider the semi-major axis for the hyperbola to be negative, then (4.20) and (4.21) will be identically applicable for both the ellipse and the hyperbola and we shall use them repeatedly later.

In addition to the fundamental parameters we have examined in this section, we will be interested in many other parameters of Keplerian trajectories. These parameters and their relation to the initial motion conditions will also be examined in succeeding sections.

# 5. MAGNITUDE OF THE FLIGHT VELOCITY

The magnitude of the flight velocity at any arbitrary point of the trajectory can easily be expressed in terms of the initial motion conditions by using the kinetic energy theorem as written in the form (3.6). Rewriting (3.6) thus:

$$(5.1) \qquad d(v^2) = -\frac{2\pi_0}{r^2}\,dr$$

we obtain after integrating

$$(5.2) \qquad v^2 = v_1^2 + 2\pi_0\left(\frac{1}{r} - \frac{1}{r_1}\right)$$

where v and r are the magnitudes of the velocity vector and the radius vector of a point at any arbitrary instant of the motion, and $v_1$ and $r_1$ are the same quantities at the initial instant chosen.

Later, the relation (5.2) will often be used, reduced to the form:

$$(5.3) \qquad v^2 = v_1^2 + 2\bar{h}_1\frac{\pi_0}{r_1}$$

where

$$(5.4) \qquad \bar{h}_1 = \frac{h_1}{r}\,; \quad h_1 = r_1 - r$$

or to the form

$$(5.5) \qquad k = \frac{k_1 + \bar{h}_1}{1 + \bar{h}_1}$$

where according to (4.1)

$$(5.6) \qquad k = \frac{rv^2}{2\pi_0}\,; \quad k_1 = \frac{r_1 v_1^2}{2\pi_0}$$

As follows from the derivation, the above formulas are valid for all kinds of trajectories.

If the trajectory of the point under consideration terminates on the surface of the earth (which we shall consider to be a sphere of radius R consistent with the assumptions of section 2 - see Fig. 3, for example) then the magnitude of the velocity at the point of intersection of the trajectory and the surface of the earth is usually of interest. Let us call this point the **incident point** and denote it by $O_R$. In what follows, we will ascribe the subscript R to all quantities at the incident point. According to (5.3), the magnitude of the velocity at the incident point of interest will equal:

$$v_R^2 = v_1^2 + 2\bar{h}_{1R}\frac{\pi_0}{r_1} \tag{5.7}$$

where

$$\bar{h}_{1R} = \frac{h_{1R}}{R}\ ; \quad h_{1R} = r_1 - R \tag{5.8}$$

and correspondingly:

$$k_R = \frac{k_1 + \bar{h}_{1R}}{1 + \bar{h}_{1R}} \tag{5.9}$$

where

$$k_R = \frac{Rv_R^2}{2\pi_0}\ . \tag{5.10}$$

## 6. FLIGHT ALTITUDE

Let us designate the altitude of a point moving over the surface of the earth h, defined by the relation

$$h = r - R \tag{6.1}$$

as the flight altitude.

For a point moving along a radial trajectory the flight altitude achieves its maximum value at the instant when the point, departing from the surface of the earth and gradually losing speed, stops completely. Letting $r_{max}$ denote the maximum distance from the center of the earth for this case, we obtain from the expression (5.2)

$$r_{max} = \frac{r_1}{1 - k_1} \tag{6.2}$$

and correspondingly

$$h_{max} = r_{max} - R \ . \tag{6.3}$$

As follows from (6.2), for $k_1 \geq 1$ the point recedes from earth toward infinity:

$$r_{max} = \infty \ .$$

As a point moves along a circular trajectory, its altitude and speed of flight remain constant and equal to a certain initial altitude and initial velocity. As a point moves along an elliptical trajectory, its flight altitude varies between a certain minimum value $h_{min}$, called the **perigee**, and a certain maximum value $h_{max}$, called the **apogee**, if the trajectory is located completely outside the limits of the earth (see Fig. 4). And in conformity with the trajectory equation (3.26):

$$\begin{aligned} h_{min} &= r_{min} - R \ ; \quad r_{min} = \frac{p}{1 + e} \\ h_{max} &= r_{max} - R \ ; \quad r_{max} = \frac{p}{1 - e} \end{aligned} \tag{6.4}$$

If the trajectory intersects the earth's surface as is shown in Fig. 3, then the flight altitude varies between zero and a maximum value which will be denoted by H in this case and will be defined exactly as above:

$$H = \frac{p}{1 - e} - R \ . \tag{6.5}$$

In this case the quantity H is often called the **trajectory altitude.**

When a point moves along a parabolic or hyperbolic trajectory, the flight altitude changes from a certain minimum value $h_{min}$ to the infinitely large $h_{max} = \infty$ if the trajectories are located completely outside the limits of the earth (see Fig. 4). And in conformity with (3.26) the value of $h_{min}$ is also determined by means of formula (6.4) in this case.

If the trajectories intersect the surface of the earth, then the flight altitude changes from zero to infinity in the case of a point moving from earth, and from infinity to zero in the case of a point moving to earth from an infinitely distant point.

## 7. ANGULAR FLIGHT RANGE

One of the convenient characteristics of point motion near earth is the flight range measured along the earth's surface from the radius vector of some one point of the trajectory $O_1$ to the radius-vector of some other point $O_2$. Letting $\Phi_{12}$ denote the angle formed by these radius vectors drawn from the center of the earth, the flight range along the earth can be characterized completely by this angle. The linear range measured along the surface of a spherical earth will here equal

$$(7.1) \qquad D_{12} = R\ \Phi_{12}\ .$$

In the following, in contrast to the linear range $D_{12}$, the angular range $\Phi_{12}$ will be called the **angular flight range.**

The determination of the angular range $\Phi_{12}$ between two trajectory points is not difficult if the coordinates of these points are given. The problem is not related just to the trajectories or the point motion under consideration. Naturally, the formulation of such a purely geometric problem as the location of two points whose coordinates are given is not of interest in the study of the kinds of motion under consideration here.

Another formulation of the problem of determining the angular range between two points of the trajectories being examined is more expedient for what follows; it is the following.

The parameters (4.2)

$$(7.2) \qquad r_1, \eta_1, v_1, \theta_1$$

which completely define the trajectory, are given at one of the trajectory points, say $O_1$. It is required to determine the angular range $\Phi_{12}$ between the point $O_1$ and a point $O_2$ of the trajectory, which has only one coordinate $r_2$ given. By giving the point $O_2$ just the one coordinate $r_2$, we have posed the problem of determining the angular range between the point $O_1$, at which the parameters (7.2) have been given, and the point of trajectory intersection with a sphere of radius $r_2$ described from the center of the earth. Hence, this problem can also be represented as a problem to determine the angular range of a point of incidence on a sphere of radius $r_2$ by means of prescribed initial conditions (7.2). The determination of the angular range here reduces to the expression of the angle $\Phi_{12}$ in terms of the initial conditions (7.2).

Evidently, in the case of point motion along a radial trajectory the angular range between any of its points is zero. The formulated problem has no solution for point motion along a circular trajectory: for $r_1 \neq r_2$ it is missing and for $r_1 = r_2$ it seems indeterminate.

In the case of an elliptical, parabolic or hyperbolic trajectory, the problem of determining the angular range can be solved as follows.

In order to simplify the subsequent calculations, let us introduce a new polar axis passing through the center of the earth and the prescribed point $O_1$, and let f denote the polar angle of the moving point, measured from the new polar axis which is connected to the prescribed point $O_1$ (see Fig. 3). Then, evidently

$$(7.3) \qquad f = \eta - \eta_1$$

and the new polar angles of the points $O_1$ and $O_2$ will equal:

$$(7.4) \qquad f_1 = 0, \; f_2 = \eta_2 - \eta_1 = \Phi_{12} \, .$$

Since the trajectory equation and all the preceding formulas have been written for any arbitrary position of the polar axis passing through the center of the earth, they remain valid even for the special direction selected above of the polar axis connected to the point $O_1$. Hence, the trajectory equation in the new polar coordinates can, as before, be described in the form of (3.21)

$$(7.5)\qquad u = U_0 - U_1 \cos f - U_2 \sin f$$

and the constants $U_0$, $U_1$, $U_2$ which enter here, can be defined as before by means of formulas of the form of (4.11):

$$(7.6)\qquad \begin{cases} U_0 = \dfrac{1}{2kr\cos^2\theta} \\ U_1 = \left(U_0 - \dfrac{1}{r}\right)\cos f - \dfrac{\tan\theta}{r}\sin f \\ U_2 = \left(U_0 - \dfrac{1}{r}\right)\sin f + \dfrac{\tan\theta}{r}\cos f \end{cases}$$

and by the parameters

$$(7.7)\qquad r_1,\ f_1 = 0,\ v_1,\ \theta_1$$

prescribed at the point $O_1$. After this, the problem of determining the angular range reduces to the determination of the point of intersection of the trajectory (7.5) with a sphere whose equation has the form

$$(7.8)\qquad u = u_2 = \frac{1}{r_2} = \text{const} .$$

It is evident that to determine the desired point of intersection, it is sufficient to substitute (7.8) into (7.5) and, by solving it with respect to f, to obtain the unknown coordinate of the point $O_2$:

$$(7.9)\qquad f = f_2 = \Phi_{12}$$

which equals the desired angular range, whereupon the formulated problem is solved.

Thus, by substituting (7.8) and (7.9) into (7.5), we obtain the initial equation for the determination of the angular range in the form

$$(7.10)\qquad u_2 = U_0 - U_1 \cos\Phi - U_2 \sin\Phi$$

where, in order to simplify the writing, it has been assumed that

$$(7.11)\qquad \Phi_{12} = \Phi .$$

Transforming to the half-angle, let us rewrite equation (7.10) thus:

$$(U_0 - u_2) - U_1\left(\cos^2 \frac{\Phi}{2} - \sin^2 \frac{\Phi}{2}\right) - 2U_2 \sin \frac{\Phi}{2} \cos \frac{\Phi}{2} = 0 .$$

Dividing both sides of the equation by $\cos^2 \frac{\Phi}{2}$, as is possible for

$$\cos^2 \frac{\Phi}{2} \neq 0 \tag{7.12}$$

we obtain

$$(U_0 - u_2)\left(1 + \tan^2 \frac{\Phi}{2}\right) - U_2\left(1 - \tan^2 \frac{\Phi}{2}\right) - 2U_2 \tan \frac{\Phi}{2} = 0$$

or

$$(U_0 - u_2 + U_2)\tan^2 \frac{\Phi}{2} - 2U_2 \tan \frac{\Phi}{2} + (U_0 - u_2 - U_1) = 0 . \tag{7.13}$$

By substituting the prescribed values (7.7) into (7.6) we obtain the following simple expressions for the constants $U_0$, $U_1$ and $U_2$:

$$U_0 = \frac{1}{2k_1 r_1 \cos^2\theta_1} ; \quad U_1 = U_0 - \frac{1}{r_1} ; \quad U_2 = \frac{\tan \theta_1}{r_1} \tag{7.14}$$

Using (7.14), let us rewrite (7.13) thus:

$$\left(2U_0 - \frac{1}{r_1} - \frac{1}{r_2}\right)\tan^2 \frac{\Phi}{2} - 2\,\frac{\tan \theta_1}{r_1} \tan \frac{\Phi}{2} + \left(\frac{1}{r_1} - \frac{1}{r_2}\right) = 0 \tag{7.15}$$

or

$$\chi_1 \tan^2 \frac{\Phi}{2} - 2 \tan \theta_1 \tan \frac{\Phi}{2} - \bar{h}_1 = 0 \tag{7.16}$$

where

$$\bar{h}_1 = \frac{h_1}{r_2} ; \quad h_1 = r_1 - r_2 ; \quad \chi_1 = \frac{1}{k_1 \cos^2\theta_1} - 2 - \bar{h}_1 \tag{7.17}$$

Solving (7.16), we obtain the desired expression for the angular range in the following form:

$$(7.18) \quad \tan\frac{\Phi}{2} = \frac{\tan\theta_1 \pm \sqrt{\tan^2\theta_1 + \chi_1\bar{h}_1}}{\chi_1}$$

Let us examine the solution just obtained. It shows that in the general case, depending on the magnitude of the radius $r_2$ of the sphere, we can expect the presence either of two points of intersection of the trajectory with the sphere or of one tangent point or of no common points. Let us first determine the radius of that sphere which is tangent to the trajectory. To do this, it is sufficient to equate the discriminant of our quadratic equation to zero. Hence, to shorten the computations, we return to the equation written in the form (7.13) and obtain the desired relation as

$$(7.19) \quad (u_2 - U_0)^2 = U_1^2 + U_2^2$$

from which it follows that

$$u_2 = U_0 \pm \sqrt{U_1^2 + U_2^2}$$

or

$$\frac{1}{u_2} = \frac{\frac{1}{U_0}}{1 \pm \sqrt{\frac{U_1^2 + U_2^2}{U_0^2}}}$$

or in conformity with (4.12) and (4.13)

$$(7.20) \quad r_2 = \frac{p}{1 \pm e}$$

By comparing (7.20) with (6.4), the deduction can be made that only those spheres which pass through the vertex can be tangent to the trajectory.

In particular, an elliptical trajectory has two vertices corresponding to the values $r_{min}$ and $r_{max}$; therefore, two spheres with the radii

$$(7.21) \quad r_2 = r_{min} = \frac{p}{1 + e} \text{ and } r_2 = r_{max} = \frac{p}{1 - e}$$

can be tangent to it.

The parabolic and hyperbolic trajectories have just one vertex corresponding to the value $r_{min}$; hence, just one sphere of radius

$$(7.22) \quad r_2 = r_{min} = \frac{p}{1 + e}$$

can be tangent to each of these trajectories.

After the spheres which are tangent to the trajectories are determined, the rest are easily determined. Thus, it is evident that the elliptical trajectory will be intersected in two points by all spheres whose radii lie in the interval

$$(7.23) \quad r_{min} < r_2 < r_{max}$$

and will not be intersected by any spheres whose radii satisfy the following inequalities:

$$r_2 < r_{min} \text{ and } r_2 > r_{max}$$

The parabolic and hyperbolic trajectories will be intersected at two points by all spheres whose radius is greater than the radius of the tangent sphere

$$(7.25) \quad r_2 > r_{min}$$

and will not be intersected by any spheres whose radius is less than the radius of the tangent sphere

$$(7.26) \quad r_2 < r_{min} \, .$$

Let us now determine the angular range of the tangent points. On the basis of (7.18), the equality

$$(7.27) \quad \tan^2\theta_1 + \chi_1\bar{h}_1 = 0$$

can be written and the simple expression

$$(7.28) \quad \chi_1 = -\frac{\tan^2\theta_1}{\bar{h}_1}$$

follows for $\chi_1$. Using (7.27) and (7.28) we can reduce (7.18) to the simple form

$$(7.29) \quad \tan\frac{\Phi_F}{2} = -\frac{\bar{h}_1}{\tan\theta_1}$$

where $\bar{h}_1$ is not arbitrary but equals

$$(7.30) \quad \bar{h}_1 = \frac{r_1 - r_{min}}{r_{min}} \text{ or } \bar{h}_1 = \frac{r_1 - r_{max}}{r_{max}}$$

for the elliptical trajectory and

$$(7.31) \quad \bar{h}_1 = \frac{r_1 - r_{min}}{r_{min}}$$

for the parabolic and hyperbolic trajectories.

Moreover, since the tangent points are the vertices of our trajectories and lie on the focal axis, then (4.16), which determines the position of the focal axis relative to the arbitrarily selected polar axis, can be used to determine their angular range. The values of $U_1$ and $U_2$ are determined by (7.14) for the selected polar axis passing through the given point $O_1$. Substituting them into (4.16), we obtain the following expression for the angular range between the point $O_1$ and points lying on that half of the focal axis on which the second focus of the elliptical trajectory and the infinite point of the parabolic and hyperbolic trajectories lie:

$$(7.32) \quad \tan\Phi_F = \frac{2k_1\tan\theta_1}{1 - 2k_1 + \tan^2\theta_1}$$

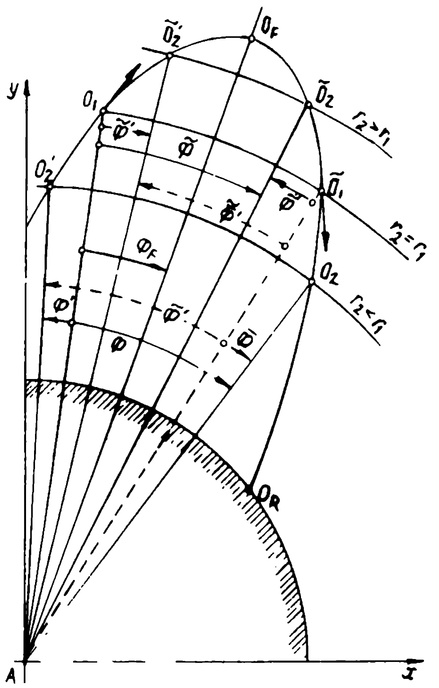

Fig. 5

To determine the angular range of the points of intersection of the elliptical trajectory with the sphere, let us use Fig. 5. Let us first examine the case when

$$(7.33)\qquad \tan\theta_1 > 0 \text{ and } r_1 > r_2,$$

that is, when the point $O_1$ is on the ascending branch of the trajectory above the prescribed sphere. One of the two existing points of intersection in this case can be considered a point of incidence on the sphere since it lies on the path of the motion of the point under consideration and it can actually be reached for motion from the prescribed point $O_1$. Let us denote this point of intersection by $O_2$. It is possible to consider another point as actually achievable for the motion of the considered point from the prescribed point $O_1$. Let it be denoted by O. Correspondingly, let $\Phi$ denote the angular range to the point $O_2$ and $\Phi'$ to the point $O_2'$. Then, by taking account of the positiveness of $\tan\theta_1$ for the case considered and that

$$(7.34)\qquad \Phi' < 0,\ \Phi > 0,\ |\Phi| > |\Phi'|,$$

we obtain

$$(7.35)\qquad \tan\frac{\Phi}{2} = \frac{\tan\theta_1 + \sqrt{\tan^2\theta_1 + \chi_1\bar{h}_1}}{\chi_1}$$

$$(7.36)\qquad \tan\frac{\Phi'}{2} = \frac{\tan\theta_1 - \sqrt{\tan^2\theta_1 + \chi_1\bar{h}_1}}{\chi_1}$$

For the case when

$$(7.37)\qquad \tan\theta_1 > 0 \text{ and } r_1 < r_2$$

we will have one of the two points of intersection more distant from the prescribed point $O_1$. Let us denote it by $\overline{O}_2$ and the range to it by $\overline{\Phi}$. Let $\overline{O}_2'$ denote the other, less distant point and $\overline{\Phi}'$ the range to it. Then, taking into account, as in the preceding case, the positiveness of $\tan \theta_1$ and that for the new case

$$(7.38) \qquad \overline{\Phi} > \overline{\Phi}' > 0$$

we obtain the same formulas (7.35) and (7.36) to determine $\overline{\Phi}$ and $\overline{\Phi}'$.

For the case when

$$(7.39) \qquad \tan \theta_1 < 0 \text{ and } r_1 > r_2 ,$$

that is, when the point $O_1$ is on the descending branch of the trajectory (in Fig. 5 it is denoted by $\overline{O}_1$ in contrast to the previous position), we will have the same points of intersection $O_2$ and $O_2'$ as in the first case. Denoting the angular range to the point $O_2$ by $\overline{\Phi}$ and to the point $O_2'$ by $\overline{\Phi}'$ and taking account of the negativeness of $\tan \theta_1$ and that for the case under consideration

$$(7.40) \qquad \overline{\Phi}' < 0, \quad \overline{\Phi} > 0, \quad |\overline{\Phi}'| > |\overline{\Phi}|$$

we obtain the same formulas (7.35) and (7.36) to determine $\overline{\Phi}$ and $\overline{\Phi}'$.

Finally, for the case when

$$(7.41) \qquad \tan \theta_1 < 0 \text{ and } r_1 < r_2$$

we will have the points of intersection $\tilde{O}_2$ and $\tilde{O}_2'$. Letting $\tilde{\overline{\Phi}}$ denote the angular range to the point $\tilde{O}_2$ and $\tilde{\overline{\Phi}}'$ the angular range to the point $\tilde{O}_2'$ and taking account of the negativeness of $\tan \theta_1$ and that for the case considered

$$(7.42) \qquad \tilde{\overline{\Phi}} < 0, \quad \tilde{\overline{\Phi}}' < 0, \quad |\tilde{\overline{\Phi}}'| > |\tilde{\overline{\Phi}}|$$

we obtain the same formulas (7.35) and (7.36) to determine $\tilde{\overline{\Phi}}$ and $\tilde{\overline{\Phi}}'$.

The following deduction can be made as a result of the analysis carried out.

For each pair of points of intersection of the elliptical trajectory with the given sphere which are symmetric relative to the focal axis, the angular range determined by (7.35) corresponds to a point reached by moving away from the axis in the motion direction, and the angular range determined by (7.36) corresponds to a second point reached by moving away from the focal axis in a direction opposite to the motion.

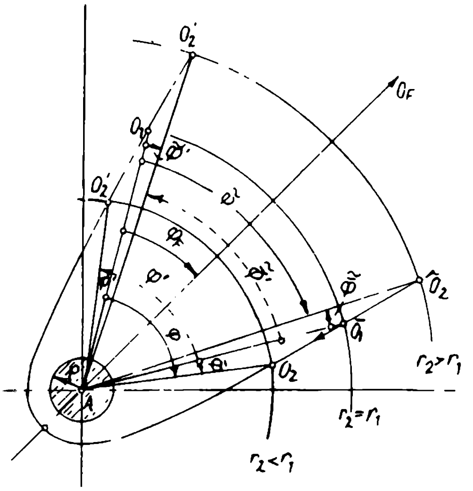

Fig. 6

For the parabolic and hyperbolic trajectories cases, perfectly analogous considerations apply and an analogous conclusion is obtained. However, since the vertex analogous to the vertex $O_F$ of the elliptic trajectory recedes to infinity for these trajectories and the motion from one point of intersection with the sphere to the other, symmetric point of intersection can only be realized via the vertex $O_F'$ (Fig. 6); then the motion over these trajectories comparable to the motion over the elliptical trajectories considered above seems to be reversed with respect to the points in which we are interested. The motion is directed from the point $O_1$ to the point $\bar{O}_1$ on the elliptical trajectories and from the point $\bar{O}_1$ to $O_1$ on the parabolic and elliptical trajectories. Hence, for each pair of intersection points of the parabolic or hyperbolic trajectories with the given sphere which are symmetric relative to the focal axis, the angular range determined by (7.36) corresponds to the point receding from the axis in the direction of motion, and the angular range determined by (7.35) corresponds to the second point receding from the focal axis in the direction opposite to the motion.

In the sequel, we shall be interested in those points of intersection of the trajectory with the sphere which can be achieved by the moving point as it moves further from the point $O_1$. For brevity, let these points be designated points of intersection lying in the direction of motion or, simply, forward-lying points of intersection and let the rest be designated backward-lying points of intersection.

For elliptical trajectories over which the moving point can be displaced and can complete full revolutions without encountering any obstacle in its path, both points of intersection with the sphere are forward-lying. Here, in the case $r_1 > r_2$ the point $O_2$ will always be first in the path of the motion, followed by the point $O_2'$ (see Fig. 5). Hence, it is necessary to use (7.35) when determining the angular range to the first point in this case, and, therefore, (7.36) when determining the angular range to the second point. In the case $r_1 < r_2$, the point $\tilde{O}_2'$ will always be first in the motion path and $\tilde{O}_2$ will be second. In this case it is necessary to use (7.36) when determining the angular range to the first point and (7.35) when determining the angular range to the second point.

For parabolic and hyperbolic trajectories in the case $r_1 > r_2$, both points of intersection are forward-lying for $\theta_1 < 0$ and backward-lying for $\theta_1 > 0$. Hence, the point $O_2$ will always be the first in the motion path among the forward-lying points and the point $O_2'$ the second (see Fig. 6). Consequently, (7.35) must be used in determining the angular range to the first point and (7.36) to the second point, just as for the elliptical trajectories. We shall not investigate the determination of the angular range to the backward-lying points. In the case $r_1 < r_2$ just one point, namely $\tilde{O}_2'$, will always be forward-lying and the other, $\tilde{O}_2$, will be backward-lying, and the angular range to the forward-lying point will be determined by (7.36).

**Hence, in the case $r_1 > r_2$ for all trajectories, the angular range to the first of the forward-lying points of intersection is determined by (7.35) and in the case $r_1 < r_2$ the angular range to the first of the forward-lying points is determined by (7.36).**

## 8. DIRECTION OF THE FLIGHT VELOCITY

The direction of the velocity at any point of the trajectory determined by the angle $\theta$ can be related to the initial conditions (7.2) prescribed at some point $O_1$, by means of the following simple relation:

$$\sigma = r_1 v_1 \cos \theta_1 = rv \cos \theta \tag{8.1}$$

which results from (4.9).

In those cases when it is desirable to have a relation between the angles $\theta_1$ and $\theta$ in terms of the tangents rather than in terms of the cosines, the desired dependence can be obtained as follows. Using (4.12), let us write

$$p = 2k_1r_1\cos^2\theta_1 = 2kr\cos^2\theta$$

or

$$\frac{k_1r_1}{\cos^2\theta} = \frac{kr}{\cos^2\theta_1} \tag{8.2}$$

It is here evident that the cases of the radial and circular trajectories for which, respectively,

$$\cos^2\theta \equiv 0 \quad \text{and} \quad \cos^2\theta \equiv 1 \tag{8.3}$$

are eliminated from the considerations.

Substituting (5.5) into (8.2), we obtain

$$\frac{k_1r_1}{\cos^2\theta} = \frac{k_1 + \bar{h}_1}{1 + \bar{h}_1}\frac{r}{\cos^2\theta_1} .$$

Making the substitution according to (5.4)

$$1 + \bar{h}_1 = \frac{r_1}{r} \tag{8.4}$$

we can write the last equality as

$$\frac{k_1r_1}{\cos^2\theta} = \frac{k_1r^2}{r_1\cos^2\theta_1} + \frac{\bar{h}_1r^2}{r_1\cos^2\theta_1}$$

or

$$\frac{r_1^2}{r^2}\frac{1}{\cos^2\theta} = \frac{1}{\cos^2\theta_1} + \frac{\bar{h}_1}{k_1\cos^2\theta_1}$$

or

$$\frac{r_1^2}{r^2}\tan^2\theta = \tan^2\theta_1 + 1 - \frac{r_1^2}{r^2} + \frac{\bar{h}_1}{k_1\cos^2\theta_1} .$$

Replacing $r_1^2/r^2$ in the right side by the expression resulting from (8.4), we obtain

$$\frac{r_1^2}{r^2}\tan^2\theta = \tan^2\theta_1 + 1 - (1 + \bar{h}_1)^2 + \frac{\bar{h}_1}{k_1\cos^2\theta_1}$$

$$= \tan^2\theta_1 + \bar{h}_1\left(\frac{1}{k_1\cos^2\theta_1} - 2 - \bar{h}_1\right)$$

Using (7.17) we can write

$$(8.5) \qquad \frac{r_1^2}{r^2}\tan^2\theta = \tan^2\theta_1 + \chi_1\bar{h}_1 .$$

From (7.18) it follows that

$$(8.6) \qquad \pm\sqrt{\tan^2\theta_1 + \chi_1\bar{h}_1} = \chi_1\tan\frac{\Phi}{2} - \tan\theta_1 .$$

Hence, as is seen from (7.35) and (7.36), in order to obtain the positive root, it is necessary to substitute the value of $\Phi$ corresponding to the points $O_2$ and $\tilde{O}_2$ in the right side of (8.6) and in order to obtain the negative root, to substitute the value of $\Phi$ corresponding to the points $O_2'$ and $\tilde{O}_2'$ of the trajectory (see Fig. 5). Also taking into account the fact that $\tan\theta < 0$ at the points $O_2$ and $\tilde{O}_2$ of the trajectory and $\tan\theta > 0$ at the points $O_2'$ and $\tilde{O}_2'$, we can rewrite (8.5) for $\tan\theta$ thus:

$$(8.7) \qquad -\tan\theta = \frac{r}{r_1}\left(\chi_1\tan\frac{\Phi}{2} - \tan\theta_1\right) .$$

On the basis of (7.16), we can write the equality

$$(8.8) \qquad \chi_1 \tan\frac{\Phi}{2} - \tan\theta_1 = \tan\theta_1 + \bar{h}_1 \cot\frac{\Phi}{2}$$

which is valid for both $\Phi$ and for $\Phi'$. Using it, let us rewrite (8.7) in the final form

$$(8.9) \qquad \tan\theta = -\frac{1}{1+\bar{h}_1}\left(\tan\theta_1 + \bar{h}_1 \cot\frac{\Phi}{2}\right).$$

If it is necessary to determine the direction of the velocity at the point of intersection of an elliptical trajectory with a sphere of radius $r_1$, then

$$(8.10) \qquad \tan\theta_2 = -\frac{1}{1+\bar{h}_1}\left(\tan\theta_1 + \bar{h}_1 \cot\frac{\Phi}{2}\right)$$

can be used in all cases. However, it is here necessary to take the value of $\Phi$ determined from (7.35) for the points $O_2$ and $\tilde{O}_2$ and the value of $\Phi'$ determined from (7.36) for the points $O_2'$ and $\tilde{O}_2'$.

Evidently this derivation is valid for all three kinds of trajectories, elliptical, parabolic, and hyperbolic.

## 9. FLIGHT TIME

The determination of the flight time is related to the integration of the differential dependence expressing the law of the motion of the point along its trajectory.

1. For a radial trajectory, the dependence

$$\frac{dr}{dt} = \pm v$$

can be taken as such a differential dependence: hence there follows

$$(9.1) \qquad dt = \pm\frac{dr}{v}.$$

Expressing v according to (5.2) in terms of r, we obtain

$$(9.2) \qquad v^2 = \frac{b}{r} - a$$

where

$$(9.3) \qquad b = 2\pi_0, \; a = \frac{2\pi_0}{r_1}(1 - k_1) = \frac{2\pi_0}{r}(1 - k) \; .$$

In order to simplify the operation of integrating the dependence (9.1) it is convenient to transform from the variable r to the variable v rather than from v to r by means of the relations:

$$(9.4) \qquad \frac{b}{r} = a + v^2 \; ; \quad dr = - \frac{2bv}{(a + v^2)^2} \, dv$$

which are easily obtained from (9.2). As a result, (9.1) can be transformed to the following:

$$(9.5) \qquad dt = \mp \frac{2b \, dv}{(a + v^2)^2} \; .$$

Let $O_F$ denote the point of maximum distance from the earth's surface on a radial trajectory by analogy with the corresponding point of an elliptical trajectory.

For radial trajectories for which the point $O_F$ lies at a finite range, the flight time from the point $O_1$ to the point $O_2$ can be measured by two methods: as the flight time from the point $O_1$ to the point $O_2$ along the shortest path, and as the flight time from the point $O_1$ to the point $O_F$ and from the point $O_F$ to the point $O_2$. Let $T'_{12}$ denote the first flight time and $T_{12}$ the second. Then

$$(9.6) \qquad T_{12} = - \int_{v_1}^{0} \frac{2b \, dv}{(a + v^2)^2} + \int_{0}^{v_2} \frac{2b \, dv}{(a + v^2)^2}$$

and

$$
\begin{aligned}
T'_{12} &= -\int_{v_1}^{v_2} \frac{2b\,dv}{(a + v^2)^2} \qquad \text{for } v_1 > v_2 \\
T'_{12} &= \int_{v_1}^{v_2} \frac{2b\,dv}{(a + v^2)^2} \qquad \text{for } v_1 < v_2
\end{aligned}
\tag{9.7}
$$

since the velocity diminishes from $v_1$ to zero for motion from the point $O_1$ to the point $O_F$ and increases from zero to $v_2$ for motion from the point $O_F$ to $O_2$.

For radial trajectories for which the point $O_F$ is at infinity, the flight time between two points can be measured by only one method, as the flight time from the point $O_1$ to the point $O_2$ along the shortest path; therefore, it will be determined by (9.7).

The integral

$$I = 2b \int \frac{dv}{(a + v^2)^2} \tag{9.8}$$

figures in (9.6) and (9.7) and it can be represented in the form:

$$
\begin{aligned}
I &= \frac{2b}{a} \int \frac{a + v^2 - v^2}{(a + v^2)^2}\,dv = \frac{2b}{a}\left[\int \frac{dv}{a + v^2} - \int \frac{v^2\,dv}{(a + v^2)^2}\right] \\
&= \frac{2b}{a}\left[\int \frac{dv}{a + v^2} + \frac{1}{2}\int v\; d\left(\frac{1}{a + v^2}\right)\right] \\
&= \frac{2b}{a}\left[\int \frac{dv}{a + v^2} + \frac{1}{2}\,\frac{v}{a + v^2} - \frac{1}{2}\int \frac{dv}{a + v^2}\right]
\end{aligned}
$$

or

$$I = \frac{b}{a}\left[\frac{v}{a + v^2} + \int \frac{dv}{a + v^2}\right]. \tag{9.9}$$

For radial trajectories in which the point $O_F$ lies at a finite range, $k < 1$; therefore, in conformity with (9.3), $a > 0$ always. Hence, the integral (9.9) can be reduced for them to the form

$$(9.10)\qquad I = \frac{b}{a}\left[\frac{v}{a + v^2} + \frac{1}{\sqrt{a}} \arctan \frac{v}{\sqrt{a}}\right] .$$

Moreover, in conformity with (6.2) and (9.3), the quantity $a$ for these trajectories can be represented thus:

$$(9.11)\qquad a = \frac{2\pi_0}{r_{max}} ; \quad r_{max} = \frac{r_1}{1 - k_1} = \frac{r}{1 - k} .$$

Using (9.11), (9.3) and (9.4), let us write (9.10) as follows:

$$(9.12)\qquad I = \frac{r_{max}}{v}\left[k + \sqrt{\frac{k}{1 - k}} \arctan \sqrt{\frac{k}{1 - k}}\right] .$$

Now (9.6) and (9.7) can be reduced to the form:

$$(9.13)\qquad T_{12} = \left[\frac{r_{max}}{v_1}\left(k_1 + \sqrt{\frac{k_1}{1 - k_1}} \arctan \sqrt{\frac{k_1}{1 - k_1}}\right) + \frac{r_{max}}{v_2}\left(k_2 + \sqrt{\frac{k_2}{1 - k_2}} \arctan \sqrt{\frac{k_2}{1 - k_2}}\right)\right]$$

$$(9.14)\qquad T'_{12} = \pm\left[\frac{r_{max}}{v_1}\left(k_1 + \sqrt{\frac{k_1}{1 - k_1}} \arctan \sqrt{\frac{k_1}{1 - k_1}}\right) - \frac{r_{max}}{v_2}\left(k_2 + \sqrt{\frac{k_2}{1 - k_2}} \arctan \sqrt{\frac{k_2}{1 - k_2}}\right)\right]$$

where the plus sign in the last expression corresponds to the case $v_1 > v_2$ and the minus sign to the case $v_1 < v_2$ .

For radial trajectories for which the point $O_F$ is at infinity, $k \geq 1$ and, therefore, $a \leq 0$ always, in conformity with (9.3). In the case $a = 0$, the integral (9.8) equals

$$(9.15)\qquad I = -\frac{2}{3}\frac{b}{v^2} = -\frac{2}{3}\frac{r}{v}\frac{1}{k} = -\frac{2}{3}\frac{r}{v}.$$

In the case $a < 0$, the expression (9.9) can be reduced to the form:

$$I = \frac{b}{a}\left[\frac{v}{v^2 + a} + \frac{1}{2\sqrt{-a}}\int\frac{(v + \sqrt{-a}) - (v - \sqrt{-a})}{v^2 + a}\,dv\right]$$

$$= \frac{b}{a}\left[\frac{v}{v^2 + a} + \frac{1}{2\sqrt{-a}}\left\{\int\frac{d(v - \sqrt{-a})}{v - \sqrt{-a}} - \int\frac{d(v + \sqrt{-a})}{v + \sqrt{-a}}\right\}\right]$$

or

$$(9.16)\qquad I = \frac{b}{a}\left(\frac{v}{v^2 + a} + \frac{1}{2\sqrt{-a}}\ln\frac{v - \sqrt{-a}}{v + \sqrt{-a}}\right).$$

Using relations (9.3) and (9.4) as in the previous case, let us write (9.16) as

$$(9.17)\qquad I = \frac{r}{v}\frac{1}{1 - k}\left(k + \frac{1}{2}\sqrt{\frac{k}{k - 1}}\ln\frac{\sqrt{\frac{k}{k - 1}} - 1}{\sqrt{\frac{k}{k - 1}} + 1}\right)$$

after which the flight time between two points of radial trajectories for which $k = 1$ is expressed as

$$(9.18)\qquad T_{12}' = \pm\frac{2}{3}\left(\frac{r_2}{v_2} - \frac{r_1}{v_1}\right)$$

and the flight time between two points of radial trajectories for which $k > 1$ is determined by the following expression:

(9.19) $$T'_{12} = \pm\left[\frac{r_2}{v_2}\,\frac{1}{k_2 - 1}\left(k_2 + \frac{1}{2}\sqrt{\frac{k_2}{k_2 - 1}}\,\ln\frac{\sqrt{\frac{k_2}{k_2 - 1}} - 1}{\sqrt{\frac{k_2}{k_2 - 1}} + 1}\right)\right.$$

$$\left. - \frac{r_1}{v_1}\,\frac{1}{k_1 - 1}\left(k_1 + \frac{1}{2}\sqrt{\frac{k_1}{k_1 - 1}}\,\ln\frac{\sqrt{\frac{k_1}{k_1 - 1}} - 1}{\sqrt{\frac{k_1}{k_1 - 1}} + 1}\right)\right]$$

The plus sign in the written expressions corresponds to the case $v_1 > v_2$ and the minus sign to the case $v_1 < v_2$.

2. For the circular trajectory it is convenient to take the dependence (3.3) as the differential dependence from which the flight time can be determined simply. Rewriting it in the form

(9.20) $$dt = \frac{r^2}{\sigma}\,d\eta$$

we obtain a very simple expression for the flight time between two points of a circular trajectory

(9.21) $$T_{12} = \frac{r^2}{\sigma}(\eta_2 - \eta_1) = \frac{r^2}{\sigma}\,\Phi_{12} = \frac{r}{v}\,\Phi_{12}\ .$$

Hence, the time of a complete revolution over a circular trajectory is

(9.22) $$T_c = \frac{2\pi r^2}{\sigma} = \frac{2\pi r}{v}\ .$$

3. For an elliptical trajectory it is also convenient to take the dependence (9.20) as the differential dependence from which the flight time can be simply determined. However, it seems to be more complicated to integrate than in the previous case, since the dependence relating $r$ and $\eta$ is more complex for the elliptical trajectory. This dependence is expressed most simply by means of (3.26). Using it, we can express $r^2$ in (9.20) in terms of $\eta$ and we can integrate with respect to the variable $\eta$, or we can express $d\eta$ in terms of $r$ and $dr$ and integrate with respect to $r$.

Let us first proceed according to the second path. Solving (3.26) for $\eta$ we obtain

$$(9.23)\qquad \eta = \eta_F \pm \arccos \frac{1}{e}\left(1 - \frac{p}{r}\right)$$

from which

$$(9.24)\qquad d\eta = \pm \frac{p}{r\sqrt{A(r)}}\, dr$$

where

$$(9.25)\qquad \sqrt{A(r)} = \sqrt{-(1 - e^2)r^2 + 2pr - p^2}$$

For elliptical trajectories $(1 - e^2) > 0$ ; hence, it is convenient to transform (9.25) as follows:

$$\sqrt{A(r)} = \sqrt{-\left[(1 - e^2)r^2 - 2pr + \frac{p^2}{1 - e^2}\right] + \frac{p^2}{1 - e^2} - p^2}$$

$$= \sqrt{\frac{p^2e^2}{1 - e^2} - \left[r\sqrt{1 - e^2} - \frac{p}{\sqrt{1 - e^2}}\right]^2}$$

$$= \frac{pe}{\sqrt{1 - e^2}} \sqrt{1 - \frac{\left[(1 - e^2)r - p\right]^2}{p^2e^2}}$$

After this, (9.24) can be written thus:

$$(9.26)\qquad d\eta = \mp \frac{\sqrt{1 - e^2}}{e} \cdot \frac{1}{\sqrt{1 - x^2}} \frac{dr}{r}$$

where

$$(9.27)\qquad x = \frac{(1 - e^2)r - p}{pe} .$$

Substituting (9.26) into (9.20), we obtain

$$(9.28)\qquad dt = \mp \frac{1}{\sigma} \frac{\sqrt{1 - e^2}}{e} \frac{r\ dr}{\sqrt{1 - x^2}} .$$

It is evidently most convenient to integrate the obtained dependence with respect to the variable x. To do this, let us express r and dr in terms of x. From (9.27) we obtain

$$r = \frac{p}{1 - e^2}(1 + ex) \tag{9.29}$$

$$dr = \frac{pe}{1 - e^2} dx \ . \tag{9.30}$$

Using the last relations, we can rewrite (9.28) thus:

$$dt = \mp \frac{p^2}{\sigma(1 - e^2)^{3/2}} \frac{1 + ex}{\sqrt{1 - x^2}} dx \ . \tag{9.31}$$

Integration of (9.31) is not difficult. It is necessary only to establish the limits of integration first. We can rewrite (9.26) as

$$\frac{dr}{d\eta} = \mp \frac{re\sqrt{1 - x^2}}{\sqrt{1 - e^2}} \ . \tag{9.32}$$

It follows from (9.32) that the sign in front of the square root when integrating (9.31) must be chosen with consideration of the sign of the derivative $dr/d\eta$ on that section of the trajectory over which the integration is made. Let us call the part of the trajectory on which $dr/d\eta > 0$ the ascending branch and the other part of the trajectory on which $dr/d\eta < 0$ the descending branch. Evidently the points at which $dr/d\eta = 0$ are the vertices of the elliptical trajectory $O_F$ and $O_F'$ and they separate the ascending and descending branches of the trajectory. It follows from (9.32) that the values of x

$$x = +1 \quad \text{and} \quad x = -1 \tag{9.33}$$

correspond to the above-mentioned points. Substituting them into (9.29), we obtain the values of r corresponding to the vertices of the trajectory:

$$r = \frac{p}{1 - e} \quad \text{and} \quad r = \frac{p}{1 + e} \tag{9.34}$$

which agree with the values of (6.4) described earlier. Here $x = +1$ corresponds to $r = r_{max}$, i.e., the point $O_F$, and $x = -1$ corresponds to $r = r_{min}$, i.e., the point $O'_F$. Evidently the plus sign must be taken on the ascending branch of the trajectory between the points $O'_F$ and $O_F$ and the minus sign on the descending branch between the same points. As in the case of the radial trajectories, we denote the time measured between the two flight points $O_1$ and $O_2$ which are on different sides of $O_F$ by $T_{12}$ and define it as the flight time from the point $O_1$ to the point $O_F$ and from $O_F$ to $O_2$. Then from (9.31) follows:

$$(9.35) \qquad T_{12} = \frac{p^2}{\sigma(1 - e^2)} \left[ \int_{x_1}^{1} \frac{(1 + ex)dx}{\sqrt{1 - x^2}} - \int_{1}^{x_2} \frac{(1 + ex)dx}{\sqrt{1 - x^2}} \right]$$

where

$$(9.36) \qquad x_1 = \frac{(1 - e^2)r_1 - p}{pe} ; \quad x_2 = \frac{(1 - e^2)r_2 - p}{pe} .$$

As before, let $T'_{12}$ denote the time measured along the shortest path between the points $O_1$ and $O_2$ on one side of the point $O_F$. On the basis of (9.31), we can then write

$$(9.37) \qquad T'_{12} = \frac{p^2}{\sigma(1 - e^2)^{3/2}} \int_{x_1}^{x_2} \frac{(1 + ex)}{\sqrt{1 - x^2}} dx \qquad \text{for } r_1 < r_2$$

$$(9.37') \qquad T'_{12} = \frac{-p^2}{\sigma(1 - e^2)^{3/2}} \int_{x_1}^{x_2} \frac{(1 + ex)}{\sqrt{1 - x^2}} dx \qquad \text{for } r_1 > r_2$$

where $x_1$ and $x_2$ are also determined by (9.36); the expression (9.37) corresponds to the case when both points are located on the ascending branch of the trajectory and (9.37') to the case when they are on the descending branch.

The time for a complete revolution on an elliptical trajectory is

$$(9.38) \qquad T_c = \frac{p^2}{\sigma(1 - e^2)^{3/2}} \left[ \int_{-1}^{+1} \frac{(1 + ex)dx}{\sqrt{1 - x^2}} - \int_{+1}^{-1} \frac{(1 + ex)dx}{\sqrt{1 - x^2}} \right] .$$

Using the fact that

$$(9.39)\quad \int \frac{1 + ex}{\sqrt{1 - x^2}}\, dx = \int \frac{dx}{\sqrt{1 - x^2}} - e \int \frac{d(1 - x^2)}{2\sqrt{1 - x^2}}$$

$$= \arcsin x - e \sqrt{1 - x^2}$$

we can represent the flight time for a complete revolution as follows:

$$(9.40)\quad T_c = 2\pi \frac{p^2}{\sigma(1 - e^2)^{3/2}}$$

after which the expressions for $T_{12}$ and $T'_{12}$ can be represented as:

$$(9.41)\quad T_{12} = T_r\left[\left(\frac{\pi}{2} - \arcsin x_1 + e\sqrt{1 - x_1^2}\right) + \left(\frac{\pi}{2} - \arcsin x_2 + e\sqrt{1 - x_2^2}\right)\right]$$

$$(9.42)\quad T'_{12} = \pm T_r\left[\left(\frac{\pi}{2} - \arcsin x_1 + e\sqrt{1 - x_1^2}\right) - \left(\frac{\pi}{2} - \arcsin x_2 + e\sqrt{1 - x_2^2}\right)\right]$$

where

$$(9.43)\quad T_r = \frac{T_c}{2\pi} = \frac{p^2}{\sigma(1 - e^2)^{3/2}}$$

is the **average time** corresponding to the rotation of a point through one radian around the center of attraction.*

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*Briefly, the **mean radian time**; in conformity with this we append the subscript r thereto.

Using expressions for $\sigma$ and a in the form:

$$(9.44) \quad \sigma = \sqrt{\pi_0 p}\,; \quad a = \frac{p}{1 - e^2}$$

we can transform the expression for $T_r$ to:

$$(9.45) \quad T_r = \frac{T_c}{2\pi} = \sqrt{\frac{a^3}{\pi_0}}\ .$$

Before reducing (9.41) and (9.42) to final form, let us integrate the dependence (9.20) with respect to the variable $\eta$ by expressing r in terms of $\eta$ by using (3.26). As a result we obtain:

$$(9.46) \quad dt = \frac{p^2}{\sigma} \frac{d\eta}{\left[1 - e \cos(\eta - \eta_F)\right]^2}$$

or

$$(9.47) \quad dt = \frac{p^2}{\sigma} \frac{d\eta}{\left[1 + e \cos(\eta - \eta'_F)\right]^2}$$

where

$$(9.48) \quad \eta'_F = \eta_F + \pi$$

is an angle which is more useful than the angle $\eta_F$ in astronomy.

Let $\alpha$ denote an angle measured from the vertex of the trajectory $O'_F$ in the motion direction; then we can write

$$(9.49) \quad \alpha = \eta - \eta'_F$$

and

$$(9.50) \quad dt = \frac{p^2}{\sigma} \frac{d\alpha}{(1 + e \cos \alpha)^2}\ .$$

Dividing numerator and denominator on the right side by $\cos^4 \alpha/2$, which is valid for all values of $\alpha$ with the exception of

(9.51) $\alpha = N\pi$

where N is any integer, we obtain

$$dt = \frac{2p^2}{\sigma} \frac{\frac{1}{\cos^2 \frac{\alpha}{2}} \frac{d\left(\frac{\alpha}{2}\right)}{\cos^2 \frac{\alpha}{2}}}{\left[-\frac{1}{\cos^2 \frac{\alpha}{2}} + e \frac{\cos \alpha}{\cos^2 \frac{\alpha}{2}}\right]^2} = \frac{2p^2}{\sigma} \frac{\left(1 + \tan^2 \frac{\alpha}{2}\right) d\left(\tan \frac{\alpha}{2}\right)}{\left[1 + \tan^2 \frac{\alpha}{2} + e\left(1 - \tan^2 \frac{\alpha}{2}\right)\right]^2}$$

$$= \frac{2p^2}{\sigma} \frac{\left(1 + \tan^2 \frac{\alpha}{2}\right) d\left(\tan \frac{\alpha}{2}\right)}{\left[(1 + e) + (1 - e)\tan^2 \frac{\alpha}{2}\right]^2} = \frac{2p^2}{\sigma(1 + e)^2} \frac{\left(1 + \tan^2 \frac{\alpha}{2}\right) d\left(\tan \frac{\alpha}{2}\right)}{\left[1 + \frac{1 - e}{1 + e} \tan^2 \frac{\alpha}{2}\right]^2}$$

Multiplying and dividing the right side by $\left|\frac{1 - e}{1 + e}\right|^{3/2} > 0$, let us rewrite the expression for dt:

$$dt = \frac{2p^2}{\sigma(1 + e)^2 \left(\frac{1 - e}{1 + e}\right)^{3/2}} \frac{\left(\frac{1 - e}{1 + e} + \frac{1 - e}{1 + e} \tan^2 \frac{\alpha}{2}\right) d\left(\sqrt{\frac{1 - e}{1 + e}} \tan \frac{\alpha}{2}\right)}{\left[1 + \frac{1 - e}{1 + e} \tan^2 \frac{\alpha}{2}\right]^2} .$$

Then, using the notation

(9.52) $\tan \frac{\epsilon}{2} = \sqrt{\frac{1 - e}{1 + e}} \tan \frac{\alpha}{2} = \sqrt{\frac{1 - e}{1 + e}} \tan \frac{\eta - \eta'_F}{2}$ , we can represent the expression for dt as follows:

$$dt = \frac{2p^2}{\sigma(1-e^2)}\sqrt{\frac{1+e}{1-e}}\;\frac{\left[\left(\frac{1-e}{1+e}-1\right)+\left(1+\tan^2\frac{\epsilon}{2}\right)\right]d\left(\tan\frac{\epsilon}{2}\right)}{\left(1+\tan^2\frac{\epsilon}{2}\right)^2}$$

$$= \frac{2p^2}{\sigma(1-e^2)}\sqrt{\frac{1+e}{1-e}}\left[-\frac{\frac{2e}{1+e}\,d\left(\tan\frac{\epsilon}{2}\right)}{\left(1+\tan^2\frac{\epsilon}{2}\right)^2}+\frac{d\left(\tan\frac{\epsilon}{2}\right)}{1+\tan^2\frac{\epsilon}{2}}\right]$$

$$= \frac{2p^2}{\sigma(1-e^2)}\sqrt{\frac{1+e}{1-e}}\left[-\frac{2e}{1+e}\cos^2\frac{\epsilon}{2}\,\frac{1}{2}d\epsilon+\frac{1}{2}d\epsilon\right]$$

$$= \frac{p^2}{\sigma(1-e^2)}\sqrt{\frac{1+e}{1-e}}\left[-\frac{e}{1+e}(1+\cos\epsilon)d\epsilon+d\epsilon\right]$$

$$= \frac{p^2}{\sigma(1-e^2)^{3/2}}(d\epsilon - e\cos\epsilon\; d\epsilon)\;.$$

Finally, using (9.43), we obtain the final expression for dt:

$$(9.53)\qquad dt = T_r\,(d\epsilon - e\cos\epsilon\; d\epsilon)$$

Integration of the dependence (9.53) is not difficult; it is only necessary to establish the limits of integration. In contrast to the preceding case, the integrand has no singularities. Hence, to establish the limits of integration it is sufficient to clarify which values of $\epsilon$ correspond to the points $O_1$ and $O_2$. Returning to (9.52), we obtain that the value of $\epsilon_1$ determined by the expression

$$(9.54)\qquad \tan\frac{\epsilon_1}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\eta_1-\eta'_F}{2}$$

corresponds to the point $O_1$, and the value of $\epsilon_2$ determined by the expression

$$(9.55)\qquad \tan\frac{\epsilon_2}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{\eta_2-\eta'_F}{2}$$

corresponds to the point $O_2$, where, in conformity with (9.48) and (4.15), the magnitude of $\eta'_F$ is determined by the formula

$$-\tan \eta'_F = \frac{(U_0 r - 1)\sin \eta + \tan \theta \cos \eta}{(U_0 r - 1)\cos \eta - \tan \theta \sin \eta}. \tag{9.56}$$

Integrating (9.53) from the point $O_1$ to the point $O_2$, we obtain an expression for the flight time between the points in the following form:

$$T_{12} = T_r \left[(\epsilon_2 - e \sin \epsilon_2) - (\epsilon_1 - e \sin \epsilon_1)\right] \tag{9.57}$$

where $\epsilon_1$ and $\epsilon_2$ are defined by (9.54) and (9.55); but (9.57) remains the same for any location of the points $O_1$ and $O_2$.

Considering the parameters $r_1$, $\eta_1$, $v_1$, $\theta_1$ prescribed at the point $O_1$ and the parameter $r_2$ at the point $O_2$, we reduce the formulas obtained for the flight time to a form convenient for computations.

Substituting (4.12) and (4.14) into (9.36), we obtain:

$$x_1 = \frac{1 - 2k_1}{e}; \quad x_2 = \frac{1 - 2k_2}{e} \tag{9.58}$$

after which (9.41) and (9.42) can be represented as follows:

$$T_{12} = T_r\left[\left(\frac{\pi}{2} - \beta_1 + \gamma_1\right) + \left(\frac{\pi}{2} - \beta_2 + \gamma_2\right)\right] \tag{9.59}$$

$$T'_{12} = \pm T_r\left[\left(\frac{\pi}{2} - \beta_1 + \gamma_1\right) - \left(\frac{\pi}{2} - \beta_2 + \gamma_2\right)\right]. \tag{9.60}$$

The plus sign in the last formula corresponds to the case $r_1 < r_2$ and the minus to the case $r_1 > r_2$ and in both formulas

$$\beta_1 = \arcsin x_1 = \arcsin \frac{1 - 2k_1}{e} \qquad \left(-\frac{\pi}{2} \le \beta_1 \le \frac{\pi}{2}\right)$$

$$\beta_2 = \arcsin x_2 = \arcsin \frac{1 - 2k_2}{e} \qquad \left(-\frac{\pi}{2} \le \beta_2 \le \frac{\pi}{2}\right)$$

$$(9.61) \quad \gamma_1 = e\sqrt{1 - x_1^2} = \sqrt{4k_1(1 - k_1)\sin^2\theta_1} = e \cos \beta_1$$

$$\gamma_2 = e\sqrt{1 - x_2^2} = \sqrt{4k_2(1 - k_2)\sin^2\theta_2} = e \cos \beta_2$$

$$T_r = \sqrt{\frac{a^2}{\pi_0}}; \quad a = \frac{r_1}{2(1 - k_1)} = \frac{r_2}{2(1 - k_2)}.$$

When using (9.57) it is convenient to transform from the angles $\eta$ to the angles f, defined by (7.3), in (9.54) and (9.55). Since (9.54) and (9.55) are valid for any case of measuring the angles $\eta$, then they are valid also for the angle f which is the angle $\eta$ measured from the radius vector to the point $O_1$. Replacing the angle $\eta$ by the angle f, we obtain

$$(9.62) \quad \begin{aligned} \tan \frac{\epsilon_1}{2} &= \sqrt{\frac{1 - e}{1 + e}}\left(- \tan \frac{\Phi'_F}{2}\right) \\ \tan \frac{\epsilon_2}{2} &= \sqrt{\frac{1 - e}{1 + e}} \tan \frac{\Phi - \Phi'_F}{2} \end{aligned}$$

where in conformity with the expression (7.32)

$$(9.63) \quad - \tan \Phi'_F = \frac{2k_1 \tan \theta_1}{1 - 2k_1 + \tan^2\theta_1}.$$

4. For the hyperbolic trajectory the expression for the flight time can be written by using the method used for the elliptical trajectory. However, in transforming the expression for $\sqrt{A(r)}$ it is necessary to take into account that $e > 1$ and to transform it thus:

$$\sqrt{A(r)} = \sqrt{(e^2 - 1)r^2 + 2pr - p^2}$$

$$= \sqrt{\left[(e^2 - 1)r^2 + 2pr + \frac{p^2}{e^2 - 1}\right] - \frac{p^2}{e^2 - 1} - p^2}$$

$$= \sqrt{\left[\sqrt{e^2 - 1}\ r + \frac{p}{\sqrt{e^2 - 1}}\right]^2 - \frac{p^2 e^2}{e^2 - 1}}$$

$$= \frac{p^2}{\sqrt{e^2 - 1}} \sqrt{\left[\frac{(e^2 - 1)\ r + p}{pe}\right]^2 - 1}\ .$$

Similarly, we can write (9.28) as

$$\text{(9.64)} \qquad dt = \mp \frac{1}{\sigma} \frac{\sqrt{e^2 - 1}}{e} \frac{r\ dr}{\sqrt{z^2 - 1}}$$

where, just as in the previous case, we can put

$$\text{(9.65)} \qquad z = \frac{(e^2 - 1)r + p}{pe}\ .$$

It follows from (9.65) that

$$\text{(9.66)} \qquad r = \frac{p}{e^2 - 1}(ez - 1)$$

and

$$\text{(9.67)} \qquad dr = \frac{pe}{e^2 - 1}\ dz\ ;$$

therefore, (9.64) can be represented as

$$\text{(9.68)} \qquad dt = \mp \frac{p^2}{\sigma(e^2 - 1)} \frac{ez - 1}{\sqrt{z^2 - 1}}\ dz\ .$$

The value $z = 1$ corresponds to the vertex of the trajectory $O'_F$ for which $r = \frac{p}{1 + e}$ ; hence, reasoning as in the preceding case, we obtain an expression for the flight time measured between two points of a hyperbolic trajectory on different sides of the vertex $O'_F$ in the form:

$$(9.69) \qquad T_{12} = \frac{p^2}{\sigma(e^2 - 1)^{3/2}} \left[ - \int_{z_1}^{1} \frac{ez - 1}{\sqrt{z^2 - 1}} dz + \int_{1}^{z_2} \frac{ez - 1}{\sqrt{z^2 - 1}} dz \right]$$

and for the flight time measured between points on one side of the vertex $O'_F$ of the trajectory, in the following form:

$$(9.70) \qquad T_{12} = \pm \frac{p^2}{\sigma(e^2 - 1)^{3/2}} \int_{z_1}^{z_2} \frac{ez - 1}{\sqrt{z^2 - 1}} dz$$

The plus sign corresponds to the case $r_1 < r_2$ and the minus sign to the case $r_1 > r_2$ (it is assumed that the motion along the trajectory is from the point $O_1$ to the point $O_2$).

Using the fact that

$$\int \frac{ez - 1}{\sqrt{z^2 - 1}} dz = e \int \frac{d(z^2 - 1)}{2\sqrt{z^2 - 1}} - \int \frac{dz}{\sqrt{z^2 - 1}} = e\sqrt{z^2 - 1} - \operatorname{arccosh} z$$

$$= e\sqrt{z^2 - 1} - \ln(z + \sqrt{z^2 - 1}) ,$$

we represent the expression for $T_{12}$ and $T'_{12}$ in the final form:

$$(9.71) \qquad \begin{aligned} T_{12} &= T_r \left[ \left( e\sqrt{z_2^2 - 1} - \operatorname{arccosh} z_2 \right) + \left( e\sqrt{z_1^2 - 1} - \operatorname{arccosh} z_1 \right) \right] \\ T'_{12} &= \pm T_r \left[ \left( e\sqrt{z_2^2 - 1} - \operatorname{arccosh} z_2 \right) - \left( e\sqrt{z_1^2 - 1} - \operatorname{arccosh} z_1 \right) \right] \end{aligned}$$

or

$$T_{12} = T_r\left[\left\{e\sqrt{z_2^2 - 1} - \ln\left(z_2 + \sqrt{z_2^2 - 1}\right)\right\}\right.$$

$$\left. + \left\{e\sqrt{z_1^2 - 1} - \ln\left(z_1 + \sqrt{z_1^2 - 1}\right)\right\}\right]$$

(9.72)

$$T'_{12} = \pm T_r\left[\left\{e\sqrt{z_2^2 - 1} - \ln\left(z_2 + \sqrt{z_2^2 - 1}\right)\right\}\right.$$

$$\left. - \left\{e\sqrt{z_1^2 - 1} - \ln\left(z_1 + \sqrt{z_1^2 - 1}\right)\right\}\right]$$

where

$$T_r = \frac{p^2}{\sigma|1 - e^2|^{3/2}}$$

$$(9.74) \quad z_1 = \frac{(e^2 - 1)r_1 + p}{pe} = \frac{2k_1 - 1}{e};$$

$$z_2 = \frac{(e^2 - 1)r_2 + p}{pe} = \frac{2k_2 - 1}{e}.$$

Just as before, going along the second path and integrating the dependence (9.46) with respect to $\eta$, taking into account the fact that $e > 1$ for a hyperbolic trajectory, we obtain instead of (9.53)

$$(9.75) \quad dt = T_r\left[\frac{e\,d\epsilon}{\cos^2\epsilon} - \frac{d\epsilon}{\cos\epsilon}\right]$$

where $\epsilon$ is defined by the expression

$$(9.76) \quad \tan\frac{\epsilon}{2} = \sqrt{\frac{e - 1}{e + 1}}\tan\frac{\omega}{2} = \sqrt{\frac{e - 1}{e + 1}}\tan\frac{\eta + \eta'_F}{2}$$

which is analogous to (9.52). Integrating (9.75) from the point $O_1$ to $O_2$, let us write the expression for the flight time between these points as follows:

$$(9.77)\quad T_{12} = T_r\left\{\left[e \tan \epsilon_2 - \ln \tan\left(\frac{\pi}{4} + \frac{\epsilon_2}{2}\right)\right] - \left[e \tan \epsilon_1 - \ln \tan\left(\frac{\pi}{4} + \frac{\epsilon_1}{2}\right)\right]\right\}$$

where $\epsilon_1$ and $\epsilon_2$ are defined by the expressions

$$(9.78)\quad \tan \frac{\epsilon_1}{2} = \sqrt{\frac{e - 1}{e + 1}}\left(- \tan \frac{\Phi'_F}{2}\right); \quad \tan \frac{\epsilon_2}{2} = \sqrt{\frac{e - 1}{e + 1}} \tan \frac{\Phi - \Phi'_F}{2}$$

which are analogous to (9.62), and the quantity $\Phi'_F$ by (9.63).

5. For a parabolic trajectory the expression for the flight time can be written also by following the method we used for the elliptical and hyperbolic trajectories. It is necessary to take account only of those simplifications, which from the very beginning, result from the equality $e = 1$ for this case.

Thus, on the basis of (9.20), (9.24), (9.25), we can write for the $e = 1$ case:

$$(9.79)\quad dt = \pm \frac{p}{\sigma} \frac{r\ dr}{\sqrt{2pr - p^2}} .$$

Exactly as in the preceding cases, the radicand vanishes at the vertex of the trajectory $O'_F$ for which

$$(9.80)\quad r = r_{min} = \frac{p}{1 + e} = \frac{p}{2} .$$

Hence, by analogy we can write the expression for the flight time measured between two points of a parabolic trajectory on either side of the vertex $O'_F$ as:

$$(9.81)\quad T_{12} = \frac{p}{\sigma}\left[- \int_{r_1}^{r_{min}} \frac{r\ dr}{\sqrt{2pr - p^2}} + \int_{r_{min}}^{r_2} \frac{r\ dr}{\sqrt{2pr - p^2}}\right]$$

and for the flight time measured between points on the same side of the vertex $O'_F$ of the trajectory as:

$$(9.82)\qquad T'_{12} = \pm \frac{p}{\sigma} \int_{r_1}^{r_2} \frac{r\ dr}{\sqrt{2pr - p^2}}$$

where the plus sign corresponds to the case $r_1 < r_2$ and the minus sign to the case $r_1 > r_2$.

Using the fact that

$$\frac{p}{\sigma} \int \frac{r\ dr}{\sqrt{2pr - p^2}} = \frac{1}{4p\sigma} \int \frac{(2pr - p^2 + p^2)\ d(2pr - p^2)}{\sqrt{2pr - p^2}}$$

$$= \frac{1}{4p\sigma}\left[\frac{2}{3}(2pr - p^2)^{3/2} + 2p^2(2pr - p^2)^{1/2}\right]$$

$$= \frac{1}{3\sigma}(r + p)\sqrt{2pr - p^2} = \frac{p^2}{3\sigma}\left(\frac{r}{p} + 1\right)\sqrt{\frac{2r}{p} - 1}$$

let us represent the expressions for $T_{12}$ and $T'_{12}$ thus:

$$(9.83)\qquad T_{12} = \frac{p^2}{3\sigma}\left[\left(\frac{r_2}{p} + 1\right)\sqrt{\frac{2r_2}{p} - 1} + \left(\frac{r_1}{p} + 1\right)\sqrt{\frac{2r_1}{p} - 1}\right]$$

$$(9.84)\qquad T'_{12} = \pm \frac{p^2}{3\sigma}\left[\left(\frac{r_2}{p} + 1\right)\sqrt{\frac{2r_2}{p} - 1} - \left(\frac{r_1}{p} + 1\right)\sqrt{\frac{2r_1}{p} - 1}\right]$$

where

$$(9.85)\qquad p = 2r\cos^2\theta\ ;\ \sigma = rv\cos\theta$$

Determining the expression for the flight time by the second means, i.e., by integrating (9.50) for the case $e = 1$, we obtain in place of (9.57)

$$T_{12} = \frac{p^2}{\sigma} \int_{\alpha_1}^{\alpha_2} \frac{d\alpha}{(1 + \cos \alpha)^2} = \frac{p^2}{\sigma} \int_{\alpha_1}^{\alpha_2} \frac{d\alpha}{4 \cos^2 \frac{\alpha}{2}}$$

$$= \frac{p^2}{2\sigma} \int_{\alpha_1}^{\alpha_2} \frac{1}{\cos^2 \frac{\alpha}{2}} \frac{d \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2}}$$

$$= \frac{p^2}{2\sigma} \int_{\alpha_1}^{\alpha_2} \left(1 + \tan^2 \frac{\alpha}{2}\right) d\left(\tan \frac{\alpha}{2}\right)$$

or

$$T_{12} = \frac{p^2}{2\sigma}\left[\left(\tan \frac{\alpha_2}{2} + \frac{1}{3} \tan^3 \frac{\alpha_2}{2}\right) - \left(\tan \frac{\alpha_1}{2} + \frac{1}{3} \tan^3 \frac{\alpha_1}{2}\right)\right] \tag{9.86}$$

where

$$\alpha_1 = - \Phi'_F \; ; \quad \alpha_2 = \Phi - \Phi'_F \tag{9.87}$$

in which $\Phi'_F$ is defined by (9.63) and

$$\Phi = \eta_2 - \eta_1 .$$

# 10. IDENTICAL VELOCITY TRAJECTORIES. MAXIMUM RANGE TRAJECTORY

Let there be prescribed the coordinates of the point $O_1$ and the magnitude of the velocity $v_1$ at this point, i.e., the quantities

$$r_1, \eta_1, v_1 \tag{10.1}$$

Then, by prescribing various launch angles $\theta_1$ we obtain a family of trajectories, to each of which corresponds its value of the angle $\theta_1$. This family of trajectories, emerging from the same point $O_1$ and corresponding to the same value of the velocity at this point, will, for brevity, be called the **family of identical-velocity trajectories**. Certain properties of identical-velocity trajectories will be examined in this section.

If we are given a certain sphere of radius $r_2$ measured from the center of the earth, then the question can be posed of the determination of points of intersection between the trajectories of the family we are interested in and the above-mentioned sphere.

Let us analyze the following cases separately in solving this problem.

1. We will first examine the case when the radius $r_2$ of the given sphere is less than the radius $r_1$ and all the trajectories of the family intersect the sphere. Evidently this case corresponds to absolute incidence of the moving point of interest onto the prescribed sphere when a certain constant velocity is imparted (to the point) at the point $O_1$ located on the sphere. Since the family of parabolic and hyperbolic trajectories passing through the point $O_1$ will always contain a trajectory going from $O_1$ to infinity, then only elliptic trajectories can be the trajectories which will absolutely intersect the prescribed sphere. Let us examine which conditions the family of elliptical identical-velocity trajectories must satisfy in order that all its trajectories should intersect the assigned sphere.

If all the elliptical trajectories of the family intersect the prescribed sphere, then the angular range $\Phi$ of the point of intersection of the trajectory with the sphere can be put into correspondence with each of them. Hence, of the two points of intersection of the ellipse with the sphere which exist in the general case and which were denoted for the case under consideration by $O_2$ and $O_2'$ on Fig. 5, let us put the angular range of the point $O_2$, defined by (7.35), into correspondence with each trajectory of the family. According to the prescribed parameters (10.1), which define the family, and the prescribed angle $\theta_1$, which determines the chosen trajectory of the family, this formula always permits finding the angular range $\Phi$ corresponding to the chosen trajectory. This same formula permits a sufficiently simple determination of the condition which the family of identical-velocity elliptical trajectories must satisfy in order for all its trajectories to intersect the given sphere. Formula (7.35) shows that there will be no intersection of the trajectory with the given sphere when the radicand is negative. Vanishing of the radicand means coincidence of two points of intersection at one point of tangency of the trajectory with the sphere. To clarify the conditions under which the radicand in (7.35) vanishes or becomes negative, let us substitute therein (7.17) for $\chi_1$. As a result, the desired condition of the presence of a point of intersection can be represented as follows:

$$(10.2) \quad \tan^2\theta_1\left(1 + \frac{\bar{h}_1}{k_1}\right) + \bar{h}_1\left(\frac{1}{k_1} - 2 - \bar{h}_1\right) > 0$$

and the condition of absence of a point of intersection as:

$$(10.3) \quad \tan^2\theta_1\left(1 + \frac{\bar{h}_1}{k_1}\right) + \bar{h}_1\left(\frac{1}{k_1} - 2 - \bar{h}_1\right) < 0 .$$

In the case under consideration, $r_1 > r_2$ and, therefore, $\bar{h}_1 > 0$ and $1 + \frac{\bar{h}_1}{k_1} > 0$ ; consequently, the inequality (10.3) can be reduced to

$$(10.4) \quad \tan^2\theta_1 < \frac{\bar{h}_1\left(2 + \bar{h}_1 - \frac{1}{k_1}\right)}{1 + \frac{\bar{h}_1}{k_1}}$$

from which it follows that under the conditions

$$(10.5) \quad 2 + \bar{h}_1 < \frac{1}{k_1} \text{ or } k_1 < \frac{1}{2 + \bar{h}_1}$$

the inequality (10.4) is impossible for any value of $\theta_1$ since the left side is always a positive quantity and the right side negative. But this in turn means that under this condition there are no trajectories among the family under consideration which would intersect the sphere.

Hence, the first case we have examined of a family of identical-velocity trajectories must be characterized by the conditions:

$$(10.6) \quad \bar{h}_1 > 0 ; \ k_1 < \frac{1}{2 + \bar{h}_1} .$$

Therefore, a definite flight range, computed by means of formula (7.35), corresponds to each of the trajectories of this family. Evidently zero angular ranges will correspond to the two trajectories of the family which correspond to the angles $\theta_1 = \pi/2$ and $\theta_1 = -\pi/2$ and, therefore, degenerate into radial trajectories. It is also clear that upon the transition from a radial trajectory corresponding to the angle $\theta_1 = \pi/2$ to elliptical trajectories for which the angle $\theta_1$ gradually diminishes while remaining close to $\pi/2$, we will obtain an increase in the angular ranges corresponding to these trajectories. In exactly the same manner, when making the transition from a radial trajectory corresponding to the angle $\theta_1 = -\pi/2$ to elliptical trajectories for which the angle $\theta_1$ gradually increases while remaining close to $-\pi/2$, we will also obtain an increase in the angular ranges corresponding to these trajectories. Consequently, it is natural to expect that an angle $\theta_1$ is found between the angles $-\pi/2$ and $\pi/2$ to which the maximum angular range will correspond.

The problem of determining this angle mathematically requires determining the value of the angle $\theta_1$ for prescribed $r_1$, $v_1$ and $r_2$ for which the function

$$\Phi = \Phi(v_1, \theta_1, r_1, r_2) \tag{10.7}$$

takes on the maximum value. The solution of this mathematical problem reduces, as is known, to looking for the roots of the equation

$$\frac{\partial\Phi(\theta_1)}{\partial\theta_1} = 0 \ . \tag{10.8}$$

To obtain the necessary partial derivative of the angular range it is simplest to return to (7.16) and, assuming $k_1$ and $\bar{h}_1$ to be constants and $\Phi$ to be a function of only $\theta_1$, to differentiate it with respect to $\theta_1$. As a result we obtain

$$\frac{\partial \chi_1}{\partial \theta_1} \tan^2\frac{\Phi}{2} + \chi_1 \frac{2\tan\frac{\Phi}{2}}{2\cos^2\frac{\Phi}{2}} \frac{\partial\Phi}{\partial\theta_1} - \frac{2\tan\frac{\Phi}{2}}{\cos^2\theta_1} - \frac{2\tan\theta_1}{2\cos^2\frac{\Phi}{2}} \frac{\partial\Phi}{\partial\theta_1} = 0$$

where on the basis of (7.17)

$$\frac{\partial \chi_1}{\partial \theta_1} = \frac{2\sin\theta_1}{k_1\cos^2\theta_1} \ .$$

Solving the obtained equation with respect to $\partial\Phi/\partial\theta_1$, we write

$$\frac{\partial\Phi}{\partial\theta_1} = \left(\frac{2\tan\theta_1}{k_1\cos^2\theta_1}\tan^2\frac{\Phi}{2} - \frac{2\tan\frac{\Phi}{2}}{\cos^2\theta_1}\right)\left(\frac{\tan\theta_1}{\cos^2\frac{\Phi}{2}} - \chi_1\frac{\tan\frac{\Phi}{2}}{\cos^2\frac{\Phi}{2}}\right)^{-1}$$

or

$$\frac{\partial\Phi}{\partial\theta_1} = \frac{2\sin\frac{\Phi}{2}\cos\frac{\Phi}{2}}{\cos^2\theta_1}\left(\frac{\tan\theta_1\tan\frac{\Phi}{2}}{k_1} - 1\right)\frac{1}{\tan\theta_1 - \chi_1\tan\frac{\Phi}{2}}$$

Using (8.8) and (8.10), let us rewrite the obtained expression as follows:

$$(10.9)\qquad \frac{\partial\Phi}{\partial\theta_1} = -\frac{\left(\tan\theta_1\tan\frac{\Phi}{2} - k_1\right)\sin\Phi}{\left(\tan\theta_1 + \bar{h}_1\cot\frac{\Phi}{2}\right)k_1\cos^2\theta_1}$$

or

$$(10.10)\qquad \frac{\partial\Phi}{\partial\theta_1} = \frac{\left(\tan\theta_1\tan\frac{\Phi}{2} - k_1\right)\sin\Phi}{(1+\bar{h}_1)\tan\theta_1\, k_1\cos^2\theta_1}\,.$$

Hence, it follows that the derivative $\partial\Phi/\partial\theta_1$ vanishes for $\Phi \neq 0$ only when

$$(10.11)\qquad k_{1m} = \tan\theta_{1m}\tan\frac{\Phi_m}{2}\,.$$

This will evidently also be the desired relation between the trajectory parameters which will guarantee maximum range. Later, just as has been done above, we shall ascribe the lower subscript to the right, m, to the trajectory parameters which guarantee maximum range, and the trajectory itself shall be called the trajectory of maximum or greatest range.

The desired thrust angle can be determined from (10.11) and we shall call it the **angle of thrust of the maximum or greatest range**; it will be denoted, as above, by $\theta_{1m}$:

$$(10.12)\quad \tan\theta_{1m} = k_{1m}\cot\frac{\Phi_m}{2} .$$

However, it is still impossible to determine the angle $\theta_{1m}$ by means of the given parameters (10.1) by using (10.12); to do this it is necessary to find $\Phi_m$.

In order to find $\Phi_m$, let us substitute (10.12) into (7.16). We obtain

$$\chi_{1m}\tan^2\frac{\Phi_m}{2} - 2k_{1m}\cot\frac{\Phi_m}{2}\tan\frac{\Phi_m}{2} - \bar{h}_1 = 0$$

where

$$\chi_{1m} = \frac{1}{k_{1m}\cos^2\theta_{1m}} - (2 + \bar{h}_1) = \frac{1 + \tan^2\theta_{1m}}{k_{1m}} - (2 + \bar{h}_1)$$

$$= \left(\frac{1}{k_{1m}} - 2 - \bar{h}_1\right) + k_{1m}\cot^2\frac{\Phi_m}{2} .$$

Hence, the equation to determine $\Phi_m$ is reduced to the form

$$(10.13)\quad \frac{1}{k_{1m}} - 2 - \bar{h}_1\tan^2\frac{\Phi_m}{2} = k_{1m} + \bar{h}_1$$

from which

$$(10.14)\quad \tan\frac{\Phi_m}{2} = \sqrt{\frac{k_{1m} + \bar{h}_1}{\frac{1}{k_{1m}} - 2 - \bar{h}_1}} .$$

Substituting (10.14) into (10.12) we obtain the formula to determine $\theta_{1m}$ for the given parameters (10.1) in the form

$$(10.15)\quad \tan\theta_{1m} = \sqrt{\frac{k_{1m} - k_{1m}^2(2 + \bar{h}_1)}{k_{1m} + \bar{h}_1}} .$$

Formula (10.14) permits a particularly graphic answer to the question: Can a point with a velocity $v_1$ at an altitude $\bar{h}_1$ move along an elliptical trajectory enclosing a sphere with the radius $r_2$ described from the center of the earth?

Evidently, an elliptical trajectory will enclose this sphere when

$$(10.16)\quad k_{1m} > \frac{1}{2 + \bar{h}_1} ,$$

i.e., when the right side of (10.14) is imaginary and, therefore, the point of intersection of the trajectory with the sphere is lacking.

We shall turn below to the consideration of a case where the maximum-range trajectory degenerates into a trajectory enclosing the prescribed sphere.

2. Thus, let us examine the case when the family of identical-velocity trajectories is characterized by the following conditions:

$$(10.17)\quad \bar{h}_1 > 0 ; \quad k_1 > \frac{1}{2 + \bar{h}_1}$$

As follows from (10.4), points of intersection with the prescribed sphere are lacking in this case for trajectories to which correspond values of $\theta_1$ satisfying the inequality

$$(10.18)\quad |\theta_1| < \theta_{1k}$$

where

$$(10.19)\quad \theta_{1k} = \arctan\sqrt{\frac{\bar{h}_1\left(2 + \bar{h}_1 - \frac{1}{k_1}\right)}{1 + \frac{\bar{h}_1}{k_1}}}$$

And points of intersection with the prescribed sphere are natural for trajectories to which correspond values of $\theta_1$ satisfying the inequality

$$(10.20)\quad |\theta_1| > \theta_{1k} .$$

Each of the latter trajectories can be put in correspondence with the angular range $\Phi$ of the point of intersection which is foremost among the forward-lying points of intersection. This angular range, as has been shown in section 7, is determined by (7.35) for all kinds of trajectories in the case $\bar{h}_1 > 0$.

For all kinds of trajectories a zero angular range will correspond to the angle $\theta_1 = -\pi/2$. As the angle $\theta_1$ increases from the value $-\pi/2$, an increase in the angular range can be expected as long as the angle $\theta_1$ is not equal to $-\theta_{1k}$, for which the point of intersection transforms into a point of tangency.* If $\theta_1 = -\theta_{1k}$ is substituted into (7.35), we obtain the following formula for the angular range of the point of tangency:

$$(10.21)\quad \tan\frac{\Phi_{1k}}{2} = \frac{+\bar{h}_1}{\tan\theta_{1k}} = +\sqrt{\frac{\bar{h}_1\left(1 + \frac{\bar{h}_1}{k_1}\right)}{2 + \bar{h}_1 - \frac{1}{k_1}}} ,$$

also valid for all kinds of trajectories. As the angle $\theta_1$ increases further from $-\theta_{1k}$ to $\theta_{1k}$, we shall obtain trajectories enclosing the prescribed sphere and not intersecting it. As the angle $\theta_1$ increases from $\theta_{1k}$ to $\pi/2$ for elliptical trajectories, we will obtain the forward-lying points of intersection, where the angular range to the foremost of these will be determined, as before, by (7.35). At the point of tangency which is characterized by the value of the angle $\theta_1 = \theta_{1k}$, this angular range will be determined by the expression

$$(10.22)\quad \tan\frac{\Phi_{2k}}{2} = -\frac{\bar{h}_1}{\tan\theta_{1k}} = -\sqrt{\frac{\bar{h}_1\left(1 + \frac{\bar{h}_1}{k_1}\right)}{2 + \bar{h}_1 - \frac{1}{k_1}}} .$$

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*The subscript "k" has been ascribed to this angle to emphasize that it corresponds to the tangency case (Russian word "kasaniya" for tangency).

As the angle $\theta_1$ increases from the value $\theta_{1k}$, a decrease in the angular range to zero for the angle $\theta_1 = \pi/2$ can be expected. As the angle $\theta_1$ increases from $\theta_{1k}$ to $\pi/2$ for the parabolic and hyperbolic trajectories, in conformity with section 7, we will obtain only the backward-lying points of intersection of the trajectory with the prescribed sphere which will not be reached by the point under consideration moving from the point $O_1$. Hence, it has no meaning to set these trajectories in correspondence with some angular range. It is more natural to consider these trajectories as not intersecting the prescribed sphere since they recede from it to infinity.

Hence, in the case under consideration, as the angle $\theta_1$ varies between $-\pi/2$ and $-\theta_{1k}$, the angular range varies between zero and the angle $\Phi_{1k}$, which is determined by (10.21) and satisfies the condition $|\Phi_{1k}| \leq \pi$.

As the angle $\theta_1$ changes from $-\theta_{1k}$ to $\theta_{1k}$, the trajectories of the family do not intersect the prescribed sphere. As the angle $\theta_1$ changes from $\theta_{1k}$ to $\pi/2$ the elliptical trajectories intersect the prescribed sphere and the angular range changes from $\Phi_{2k}$ to zero. Here $\Phi_{2k}$ is determined by (10.22) and satisfies the condition $\Phi_{2k} = 2\pi - \Phi_{1k}$. As the angle $\theta_1$ changes between $\theta_{1k}$ and $\pi/2$ the parabolic and hyperbolic trajectories do not intersect the prescribed sphere in forward-lying points.

The monotonicity of the change in the angular range in the mentioned intervals of variation of the angle $\theta_1$ is easily discerned physically and can be established mathematically by an appropriate analysis of formula (10.10) for the partial derivative $\partial\Phi/\partial\theta_1$.

3. Third, let us examine the case when the radius $r_2$ of the prescribed sphere is larger than the radius $r_1$ and all the trajectories of the family do not intersect the sphere. Evidently only elliptical trajectories can be the trajectories that do not intersect the sphere in this case. In order to determine what conditions must be satisfied by the family of identical-velocity elliptical trajectories which do not reach the prescribed sphere, let us return to inequality (10.2) which is the condition for the presence of points of intersection of the trajectories of the family with the prescribed sphere. Let us rewrite this inequality thus:

$$(10.23) \quad \tan^2\theta_1 \left(1 + \frac{\bar{h}_1}{k_1}\right) > (-\bar{h}_1)\left(\frac{1}{k_1} - 2 - \bar{h}_1\right) .$$

and taking into account that $-1 < \bar{h}_1 < 0$ always for $r_1 < r_2$, let us examine two cases separately. We first assume that

$$(10.24) \quad 1 + \frac{\bar{h}_1}{k_1} > 0, \text{ i.e., } \frac{1}{k_1} < -\frac{1}{\bar{h}_1} \text{ or } -\bar{h}_1 < k_1 .$$

Then the inequality (10.23) can be rewritten as

$$(10.25) \quad \tan^2\theta_1 > \frac{(-\bar{h}_1)}{\left(1 + \frac{\bar{h}_1}{k_1}\right)}\left(\frac{1}{k_1} - 2 - \bar{h}_1\right) .$$

Then, we assume that

$$(10.26) \quad 1 + \frac{\bar{h}_1}{k_1} < 0, \text{ i.e., } \frac{1}{k_1} > -\frac{1}{\bar{h}_1} \text{ or } -\bar{h}_1 > k_1 .$$

Inequality (10.23) can then be rewritten as follows:

$$(10.27) \quad \tan^2\theta_1 < \frac{(-\bar{h}_1)}{\left(1 + \frac{\bar{h}_1}{k_1}\right)}\left(\frac{1}{k_1} - 2 - \bar{h}_1\right)$$

It follows from the written inequalities that the condition of absence of intersections of trajectories of the family with the given sphere must be the inequalities

$$(10.28) \quad \frac{1}{k_1} > -\frac{1}{\bar{h}_1}; \quad \frac{1}{k_1} > 2 + \bar{h}_1$$

for compliance with which the left side of inequality (10.27) must always be a positive quantity and the right side, negative, which indeed means the absence of trajectories intersecting the given sphere.

Let us determine whether the second inequality of (10.28) is a consequence of the first or not. To do this, let us verify the validity of the inequality

$$-\frac{1}{\bar{h}_1} \geq 2 + \bar{h}_1 .$$

For $\bar{h}_1 < 0$ it easily reduces to the always valid inequality

$$0 \geq -(1 + \bar{h}_1)^2$$

Hence, the inequalities (10.28) can be reduced to the one:

$$k_1 < -\bar{h}_1$$

and the considered third case of the family of identical-velocity trajectories must be characterized by the following conditions:

$$(10.29) \quad -1 < \bar{h}_1 < 0 ; \quad k_1 < -\bar{h}_1 .$$

4. The last, fourth case of a family of identical-velocity trajectories which is characterized by the absolute presence of trajectories which intersect the given sphere under the condition $\bar{h}_1 < 0$ will evidently hold upon compliance with the inequalities:

$$(10.30) \quad \frac{1}{k_1} < -\frac{1}{\bar{h}_1} \text{ or } k_1 > -\bar{h}_1 .$$

Here, if the inequalities

$$(10.31) \quad \frac{1}{2 + \bar{h}_1} > k_1 > -\bar{h}_1$$

are satisfied, then the elliptical trajectories corresponding to these inequalities and angles $\theta_1$ included within the interval

$$(10.32) \quad |\theta_1| < \theta_{1k}$$

where

$$(10.33)\quad \theta_{1k} = \arctan \sqrt{\frac{\bar{h}_1\left(2 + \bar{h}_1 - \frac{1}{k_1}\right)}{1 + \frac{\bar{h}_1}{k_1}}}$$

do not intersect the given sphere, and elliptical trajectories corresponding to the same inequalities and angles $\theta_1$ included in the interval

$$(10.34)\quad |\theta_1| > \theta_{1k}$$

intersect the given sphere at two forward-lying points. The angular range to the first of the points is hence determined by formula (7.36) and the angular range to the second, by (7.35). Evidently, as in the case considered earlier, the angular range to the first point of intersection will change from zero to $\Phi_{1k}$ as the angle $\theta_1$ changes from $-\pi/2$ to $-\theta_{1k}$ and will change from $\Phi_{2k}$ to zero as the angle $\theta_1$ changes from $\theta_{1k}$ to $\pi/2$. Because of the agreement of the expressions for $\theta_{1k}$ in the present and previous cases, the expressions for $\Phi_{1k}$ and $\Phi_{2k}$ will also agree with the expressions (10.21) and (10.22) obtained in the previous case.

When the inequality

$$(10.35)\quad k_1 > \frac{1}{2 + \bar{h}_1}$$

is satisfied, all the trajectories intersect the given sphere, where the elliptical trajectories intersect the sphere in two forward-lying points, the angular ranges to which are determined by the same formulas as above and the hyperbolic and parabolic trajectories intersect the sphere in one forward-lying point, the angular range to which is determined by (7.36).

The analysis of the possible cases of the intersection of a family of identical-velocity trajectories with a given sphere can thereby be terminated. In conclusion, however, it is interesting to examine the question of the envelope of the family of trajectories under consideration.

Let us recall* that a line tangent to different curves of a family at all points is called the envelope of the family of curves. If the family is prescribed by the equation

$$(10.36)\quad F(x,\ y,\ a) = 0$$

where $x$ and $y$ are coordinates of points of the curves and the quantity $a$ is a parameter of the family, then the envelope of the family of curves is determined by the equations

$$(10.37)\quad F(x,\ y,\ a) = 0\ ;\quad \frac{\partial F}{\partial a} = F_a(x,\ y,\ a) = 0\ .$$

In our case, it is convenient to take the trajectory equation in the form (7.5) as the equation of the identical-velocity trajectories in which $u$ and $f$ are coordinates and the parameter must be the angle $\theta_1$. The rest of the parameters are constants, by definition of the family. Rewriting (7.5) in the form of (10.36), we obtain

$$(10.38)\quad F(u,\ f,\ \theta_1) = u + U_1 \cos f + U_2 \sin f - U_0 = 0$$

where $U_0$, $U_1$, $U_2$ are defined by (7.14).

In order to write the equation of the envelope, let us find the derivative $F_\theta = \partial F/\partial\theta_1$:

$$(10.39)\quad F_\theta(u,\ f,\ \theta_1) = \frac{\partial U_1}{\partial\theta_1}\cos f + \frac{\partial U_2}{\partial\theta_1}\sin f - \frac{\partial U_0}{\partial\theta_1}$$

Substituting the following expressions which are easily obtained from (7.14)

$$(10.40)\quad \frac{\partial U_0}{\partial\theta_1} = \frac{\partial U_1}{\partial\theta_1} = \frac{\tan\theta_1}{k_1 r_1 \cos^2\theta_1}\ ;\quad \frac{\partial U_2}{\partial\theta_1} = \frac{1}{r_1\cos^2\theta_1}$$

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*See, for example: V. I. Smirnov: Kurs vysshei matematiki (Course in Higher Mathematics), vol. 2, Gostekhizdat, 1951, p. 40.

into (10.39), we obtain the desired equation in the following form:

$$(10.41)\quad F_\theta(u,\ f,\ \theta_1) = \frac{\tan\theta_1}{k_1r_1\cos^2\theta_1}\cos f + \frac{1}{r_1\cos^2\theta_1}\sin f - \frac{\tan\theta_1}{k_1r_1\cos^2\theta_1} = 0$$

Equations (10.38) and (10.41) are indeed equations of the envelope. They can easily be reduced to a single equation if the parameter $\theta_1$ is expressed from (10.41) as:

$$(10.42)\quad \tan\theta_1 = \frac{k_1\sin f}{1 - \cos f} = k_1\cot\frac{f}{2}$$

and then it is substituted into (10.38). To simplify the computations, it is convenient to represent (10.38) in the form:

$$(10.43)\quad \left(u - \frac{1}{r_1}\right) + 2U_2\sin\frac{f}{2}\cos\frac{f}{2} - 2\left(U_0 - \frac{1}{r_1}\right)\sin^2\frac{f}{2} = 0\ .$$

Substituting (10.42) into (7.14), we will have

$$(10.44)\quad \begin{aligned} U_0 &= \frac{1 + k_1^2\cot^2\frac{f}{2}}{2k_1r_1} \\ U_2 &= \frac{k_1\cot\frac{f}{2}}{r_1}\ . \end{aligned}$$

After this, substituting (10.44) into (10.43), we obtain the desired equation in the following form

$$\left(u - \frac{1}{r_1}\right) + \frac{2k_1\cos^2\frac{f}{2}}{r_1} - \frac{\sin^2\frac{f}{2} + k_1^2\cos^2\frac{f}{2}}{k_1r_1} + \frac{2\sin^2\frac{f}{2}}{r_1} = 0$$

or

$$(10.45a)\quad \left(u - \frac{1}{r_1}\right) + \frac{k_1}{r_1}\cos^2\frac{f}{2} + \frac{2k_1 - 1}{k_1r_1}\sin^2\frac{f}{2} = 0.$$

Now transforming from the half to the full angles, we can write

$$(10.45b)\quad u = \frac{1 - k_1^2}{2k_1r_1}\left(1 - \frac{1 - k_1}{1 + k_1}\cos f\right)$$

or

$$(10.46)\quad r = \frac{p_0}{1 - e_0\cos f}$$

where

$$(10.47)\quad p_0 = \frac{2k_1r_1}{1 - k_1^2}\,; \quad e_0 = \frac{1 - k_1}{1 + k_1}\,.$$

It follows from the expression for $p_0$ that the envelope exists only for elliptical trajectories when $k_1 < 1$. This result could easily have been foreseen; since it is clear from physical considerations that the families of parabolic and hyperbolic trajectories cannot have envelopes.

The obtained equation for the envelope of elliptical trajectories shows that the envelope is also an ellipse whose focal axis is directed along the line $AO_1$ connecting the center of the earth $A$ with the point $O_1$ which we have selected as the initial point for the family under consideration. Hence, as should have been expected, the enveloping ellipse is located symmetrically with respect to the line $AO_1$. The semi-axes of the enveloping ellipse equal:

$$(10.48)\quad \begin{aligned} a_0 &= \frac{p_0}{1 - e_0^2} = \frac{r_1}{2}\,\frac{1 + k_1}{1 - k_1} = \frac{r_1}{2e_0} \\ b_0 &= \sqrt{p_0a_0} = r_1\,\frac{\sqrt{k_1}}{1 + k_1} \end{aligned}$$

As is known, the distance between the foci of an ellipse equals $2a_0e_0$. For the enveloping ellipse

$$(10.49)\quad 2a_0e_0 = r_1$$

and since one focus of the enveloping ellipse is at the center of the earth A, then, on the basis of (10.49), the second focus must be at the initial point of the family which we selected, $O_1$.

Different families of elliptical trajectories and, therefore, different enveloping ellipses whose foci will, however, always coincide with the points A and $O_1$, will correspond to different values of the initial velocity $v_1$. Hence, by varying $v_1$ we obtain a family of confocal enveloping ellipses whose semi-axes grow without limit as the velocity increases and approaches the parabolic velocity. Several such enveloping ellipses are shown in Fig. 7.

The enveloping ellipse has an interesting property: none of the elliptical trajectories of the family under consideration emerges beyond its limits. Consequently, with its aid it is possible to estimate graphically the properties of the elliptical trajectories issuing from the selected initial point $O_1$ and corresponding to a given value of the initial velocity $v_1$.

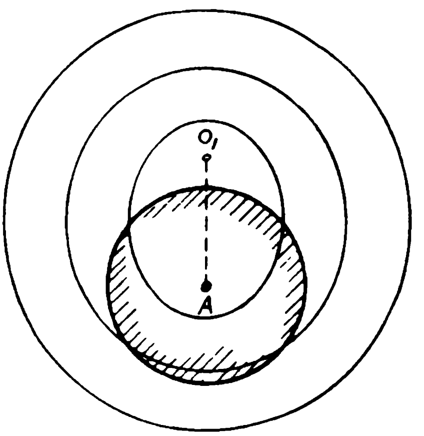

Fig. 7

For example, by determining the angular range of the point of intersection of the enveloping ellipse with a sphere of radius $r_2$ described from the center of the earth, we obtain an expression for the maximum angular range achieved by the family under consideration on the given sphere. To determine the mentioned angular range, it is sufficient to substitute the value $u = 1\ r_2$ into (10.45b) and to solve the equation thus obtained with respect to the quantity f, which in the case under consideration will also be the desired angular range $\Phi$. As a result we obtain

$$(10.50)\quad \cos \Phi_m = \frac{1 + k_1}{1 - k_1}\left[1 - \frac{2k_1}{1 - k_1^2}\,\frac{r_1}{r_2}\right] .$$

This formula does not disagree with formula (10.14) obtained earlier. Actually, if we return to the equation of the enveloping ellipse in the form (10.45a) rather than in the form (10.45b), we obtain the formula

$$(10.51) \quad \tan \frac{\Phi_m}{2} = \sqrt{\frac{k_1 + \bar{h}_1}{\frac{1}{k_1} - 2 - \bar{h}_1}}$$

which agrees completely with (10.14) and is valid both for $r_2 < r_1$ and for $r_2 > r_1$. In the $r_2 > r_1$ case the above-mentioned formula yields the maximum angular range for the second point of intersection of the elliptical trajectory with the given sphere.

## 11. IDENTICAL RANGE TRAJECTORIES. MINIMUM VELOCITY TRAJECTORY

Let a certain initial and a certain final point of the motion $O_1$ and $O_2$ be assigned. Let us determine the magnitude $v_1$ and the direction $\theta_1$ of the initial velocity which would guarantee getting from the point $O_1$ to point $O_2$.

Let us give the position of the points $O_1$ and $O_2$ in polar coordinates

$$(11.1) \quad r_1, \ f_1 = 0 \text{ and } r_2, \ f_2 = \Phi \,.$$

Then the condition for the trajectories under consideration to pass through the points $O_1$ and $O_2$ can be written in the form of (7.16) in which the quantities

$$(11.2) \quad r_1, \ r_2, \ \Phi$$

should be considered known now and the quantities $v_1$, $\theta_1$ unknown. Since there is only one equation for the determination of two unknowns, there is an infinite number of solutions. For example, it is possible to assign values of $\theta_1$ arbitrarily and to determine the appropriate value of $v_1$ for each. Hence, in this case also we obtain a family of trajectories, each of which is determined uniquely by the initial thrust angle $\theta_1$ and which passes through the assigned points $O_1$ and $O_2$. For brevity, henceforth we shall designate these trajectories as **identical-range trajectories.**

Solving (7.16) with respect to $\chi_1$ and then with respect to $k_1$, we obtain:

$$(11.3) \qquad \chi_1 = 2 \tan \theta_1 \cot \frac{\Phi}{2} + \bar{h}_1 \cot^2 \frac{\Phi}{2}$$

and

$$(11.4) \qquad \frac{1}{k_1} = \sin 2\theta_1 \cot \frac{\Phi}{2} + \cos^2\theta_1 \left(2 + \bar{h}_1 + \bar{h}_1 \cot^2 \frac{\Phi}{2}\right) ,$$

and since

$$k_1 = \frac{v_1^2 r_1}{2\pi_0}$$

then

$$(11.5) \qquad v_1^2 = \frac{2\pi_0}{r_1 \left[\sin 2\theta_1 \cot \frac{\Phi}{2} + \cos^2\theta_1 \left(2 + \bar{h}_1 + \bar{h}_1 \cot^2 \frac{\Phi}{2}\right)\right]} .$$

The obtained dependence

$$(11.6) \qquad v_1 = v_1 (\theta_1)$$

permits a determination of the magnitude of the velocity for a given direction, which is necessary to guarantee passage of the trajectory through the prescribed points $O_1$ and $O_2$. In particular, this dependence permits the determination of the minimum velocity possible for passage of the trajectory through the prescribed points. To do this it is sufficient to take the derivative of the function (11.6) with respect to $\theta_1$ and to equate it to zero. The root of the equation

$$(11.7) \qquad \frac{dv_1 (\theta_1)}{d\theta_1} = 0$$

defines the angle $\theta_{1m}$ corresponding to the desired minimum velocity. Substituting this angle into (11.5), we obtain the magnitude of the minimum velocity.

However, it is simpler to look for the maximum of $1/k_1$ rather than for the minimum $v_1$ since the found maximum of $1/k_1$ will correspond to minimum $v_1$ and the expression for $1/k_1$ is written and differentiated more simply. Thus, differentiating the right side of (11.4) with respect to $\theta_1$ and equating it to zero, we obtain

$$2 \cos 2\theta_{1m} \cot \frac{\Phi}{2} - \sin 2\theta_{1m} \left(2 + \bar{h}_1 + \bar{h}_1 \cot^2 \frac{\Phi}{2}\right) = 0$$

from which, after the left and right sides have been divided by $\cos 2\theta_{1m}$, we can determine $\theta_{1m}$ thus:

$$(11.8) \qquad \tan 2\theta_{1m} = \frac{2 \cot \frac{\Phi}{2}}{2 + \bar{h}_1 + \bar{h}_1 \cot^2 \frac{\Phi}{2}}$$

or

$$(11.9) \qquad \cot 2\theta_{1m} = \tan \frac{\Phi}{2} + \frac{\bar{h}_1}{\sin \Phi} .$$

Let us now see if the obtained value of $\theta_{1m}$ corresponds to the minimum and not the maximum velocity. To do this, we use (11.5) to determine the directions to which an infinitely large value of the magnitude of the velocity corresponds. Evidently, these directions correspond to those values of $\theta_1$ which make the denominator on the right side of (11.5) vanish. Equating this denominator to zero, we obtain

$$\cos \theta_1 \left[2 \sin \theta_1 \cot \frac{\Phi}{2} + \cos \theta_1 \left(2 + \bar{h}_1 + \bar{h}_1 \cot^2 \frac{\Phi}{2}\right)\right] = 0 ,$$

from which the desired values $\theta'_{1\infty}$ and $\theta''_{1\infty}$ can be represented as follows:

$$\cos \theta'_{1\infty} = 0 \ ; \quad \theta''_{1\infty} = \frac{\pi}{2}$$

$$\text{(11.10)} \quad -\tan \theta''_{1\infty} = \frac{2 + \bar{h}_1 + \bar{h}_1 \cot^2 \frac{\Phi}{2}}{2 \cot \frac{\Phi}{2}}$$

$$-\tan \theta''_{1\infty} = \tan \frac{\Phi}{2} + \frac{\bar{h}_1}{\sin \Phi} \ .$$

The physical meaning of the directions corresponding to the values of $\theta'_{1\infty}$ and $\theta''_{1\infty}$ obtained is the following. Evidently, for $\theta_1$ close to $\theta'_{1\infty}$ the trajectories approach a vertical, rectilinear trajectory and for $\theta_1$ close to $\theta''_{1\infty}$ the trajectories approach a rectilinear trajectory passing through the points $O_1$ and $O_2$. That the angle $\theta''_{1\infty}$ determines the direction of the line $O_1O_2$ can also be shown geometrically by comparing the expression (11.10) with the relation which can be obtained from $\Delta AO_1O_2$ (Fig. 8).

Comparing formulas (11.9) and (11.10), we can write

$$-\theta''_{1\infty} = \frac{\pi}{2} - 2\theta_{1m} = \theta'_{1\infty} - 2\theta_{1m}$$

or

$$\text{(11.11)} \quad \theta'_{1\infty} - \theta_{1m} = \theta_{1m} - \theta''_{1\infty} ; \quad \theta_{1m} = \frac{\theta'_{1\infty} + \theta''_{1\infty}}{2} \ .$$

Hence, it follows that the direction corresponding to the angle $\theta_{1m}$ is the direction of minimum initial velocity and it bisects the angles formed by the directions of the infinitely large initial velocities.

The same result can also be expressed thus: **The direction of minimum initial velocity which will guarantee incidence on the point S from the points $O_1$ and $O_2$ bisects the angle formed at the point $O_1$ by the vertical and the direction at the point of incidence $O_2$.**

The analysis of the family of identical-range elliptical trajectories is of great interest.

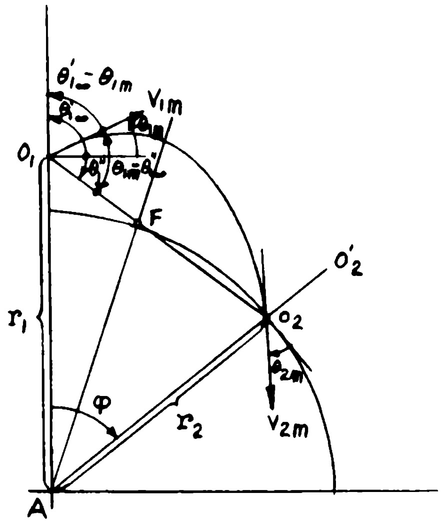

Fig. 8

Using the property of an ellipse, which is expressed as the fact that its tangent bisects the angle between the radius vector drawn from one focus to the point of tangency and the continuation of the radius vector drawn from the other focus to the point of tangency, and taking account of the result obtained above, we can arrive at the conclusion that the second focus of an elliptical trajectory corresponding to minimum initial velocity lies on the line $O_1O_2$. The position of this focus, denoted by F in Fig. 8, is determined on the line $O_1O_2$ by the following obvious relation:

$$AO_1 + FO_1 = AO_2 + FO_2$$

or

$$FO_2 - FO_1 = AO_1 - AO_2 = r_1 - r_2 = h_1$$

This relation yields a simple method of determining the position of the focus F on the line $O_1O_2$ and of constructing elliptical trajectories corresponding to a minimum initial velocity by means of the points A, $O_1$ and $O_2$.

Let us note yet another property of the elliptical trajectory under consideration, which is obtained easily by using the property, formulated above, of the tangent to the ellipse and by the fact that the focus F lies on the line $O_1O_2$. It can be formulated thus: **The tangent, at the point $O_2$, to an elliptical trajectory corresponding to a minimum initial velocity bisects the angle formed at this point by the vertical and by the direction at the point $O_1$.**

Using the last property, we can write (see Fig. 8)

$$\pi = |\theta_{2m}| + \frac{\pi}{2} + \frac{\angle O_1O_2O_2'}{2} .$$

It follows from $\Delta AO_1O_2$ that

$$\angle\, O_1O_2O_2' = \Phi + \theta_{1\infty}' + \theta_{1\infty}'' = \Phi + 2\theta_{1m} \cdot$$

As a result we obtain

$$(11.12) \quad \theta_{1m} + |\theta_{2m}| + \frac{\Phi}{2} = \frac{\pi}{2}$$

where $\theta_{2m}$ is the angle of incidence of the elliptical trajectory corresponding to the minimum initial velocity.

Finally, substituting the obtained value for $\theta_{1m}$ into (11.4), we can obtain the value of the minimum initial velocity $v_{1m}$. First, however, in order to simplify the calculations it is advantageous to use (11.9) to reduce (11.4) to the following form:

$$(11.13a) \quad \frac{1}{k_1} = \cot\frac{\Phi}{2}\,(\sin 2\theta_1 + 2\cos^2\theta_1 \cot 2\theta_{1m})$$

after which we obtain

$$\frac{1}{k_{1m}} = \cot\frac{\Phi}{2}\,(\sin 2\theta_{1m} + 2\cos^2\theta_{1m} \cot 2\theta_{1m})$$

or

$$(11.13b) \quad k_{1m} = \tan\frac{\Phi}{2}\tan\theta_{1m}\,; \quad v_{1m} = \sqrt{\frac{2\pi_0 k_{1m}}{r_1}} \,.$$

After the analysis of the properties of trajectories corresponding to minimum initial velocity, which we shall simply designate as **minimum-velocity trajectories**, let us return to the family of identical-range trajectories.

It was established above that assignment of the value of the selected parameter $\theta_1$ uniquely determined the trajectory corresponding to this value. However, it is perfectly clear that any other parameter, say, the magnitude of the initial velocity, can be chosen as the parameter of the family under consideration. In order to determine the direction $\theta_1$ of a velocity which will guarantee passage of the trajectory through the prescribed points $O_1$ and $O_2$ for a given value $v_1$ of its magnitude, it is necessary to solve relation (11.6) (obtained above between $v_1$ and $\theta_1$) for $\theta_1$. This is done most simply by returning to the above-mentioned dependence in the form (11.4). Dividing by $\cos^2\theta_1$ and multiplying by $k_1$ on the right and left sides of (11.4), we can write:

$$\frac{1}{\cos^2\theta_1} = 2k_1 \cot\frac{\Phi}{2} \tan\theta_1 + k_1 \left(2 + \bar{h}_1 + \bar{h}_1 \cot^2\frac{\Phi}{2}\right)$$

or

$$\tan^2\theta_1 - 2k_1 \cot\frac{\Phi}{2} \tan\theta_1 + 1 - k_1 \left(2 + \bar{h}_1 + \bar{h}_1 \cot^2\frac{\Phi}{2}\right) = 0\,,$$

from which

$$\tan\theta_1 = k_1 \cot\frac{\Phi}{2} \pm \sqrt{k_1^2 \cot^2\frac{\Phi}{2} + k_1 \left(2 + \bar{h}_1 + \bar{h}_1 \cot^2\frac{\Phi}{2}\right) - 1}$$

or

$$\tan\theta_1 = k_1 \cot\frac{\Phi}{2} \pm \cot\frac{\Phi}{2} \sqrt{k_1^2 + 2k_1 \tan\frac{\Phi}{2}\left(\tan\frac{\Phi}{2} \div \frac{\bar{h}_1}{\sin\Phi}\right) - \tan^2\frac{\Phi}{2}}\,. \tag{11.14}$$

The written formulas show that for each value of $k_1$ satisfying the condition

$$k_1^2 + 2k_1 \tan\frac{\Phi}{2}\left(\tan\frac{\Phi}{2} + \frac{\bar{h}_1}{\sin\Phi}\right) - \tan^2\frac{\Phi}{2} > 0 \tag{11.15}$$

there are two values of $\theta_1$ and, therefore, two trajectories corresponding to this value of $k_1$ and passing through the prescribed points $O_1$ and $O_2$. One value of $\theta_1$ and one trajectory passing through the prescribed points $O_1$ and $O_2$ correspond to the value of $k_1$ satisfying the equation

$$k_1^2 + 2k_1 \tan\frac{\Phi}{2}\left(\tan\frac{\Phi}{2} + \frac{\bar{h}_1}{\sin\Phi}\right) - \tan^2\frac{\Phi}{2} = 0 \tag{11.16}$$

Let us determine the roots of (11.16), which we shall denote by $k_{1m}$. To do this let us use (11.9) and let us rewrite (11.16) thus:

$$(11.17)\quad k_{1m}^2 + 2k_{1m}\tan\frac{\Phi}{2}\cot 2\theta_{1m} - \tan^2\frac{\Phi}{2} = 0 ,$$

after which the expression for the desired roots has the form:

$$k_{1m} = -\tan\frac{\Phi}{2}\cot 2\theta_{1m} \pm \sqrt{\tan^2\frac{\Phi}{2}\cot^2 2\theta_{1m} + \tan^2\frac{\Phi}{2}} .$$

It is not difficult to see that the value corresponding to the minus sign is superfluous since the parameter $k_1$ cannot be negative. Hence, $k_{1m}$ has just the one value

$$k_{1m} = \tan\frac{\Phi}{2}\left(-\cot 2\theta_{1m} + \sqrt{\cot^2 2\theta_{1m} + 1}\right).$$

This latter expression is easily reduced to the form

$$k_{1m} = \tan\frac{\Phi}{2}\tan\theta_{1m}$$

which shows that the found value of $k_{1m}$ agrees with the value of $k_{1m}$ determined above which corresponds to the minimum-velocity trajectory. Hence, we arrive at the conclusion that there is a unique value of $\theta_1$, which equals $\theta_{1m}$ as follows from (11.14), only for the value of $k_{1m}$ corresponding to the minimum-velocity trajectory. Therefore, a single trajectory passing through the prescribed points $O_1$ and $O_2$ corresponds only to the value $k_{1m}$ and this single trajectory is the minimum-velocity trajectory.

For each of all the other values of $k_1$ satisfying the condition

$$(11.18)\quad k_1 > k_{1m}$$

to which the condition (11.15) written above reduces, as is easy to see, there are two values of $\theta_1$, which we shall denote by $\theta_{11}$ and $\theta_{12}$ and shall define in conformity with (11.14) and (11.9) by the formulas:

$$\tan\theta_{11} = \cot\frac{\Phi}{2}\left(k_1 + \sqrt{k_1^2 + 2k_1\tan\frac{\Phi}{2}\cot 2\theta_{1m} - \tan^2\frac{\Phi}{2}}\right)$$

(11.19)

$$\tan\theta_{12} = \cot\frac{\Phi}{2}\left(k_1 - \sqrt{k_1^2 + 2k_1\tan\frac{\Phi}{2}\cot 2\theta_{1m} - \tan^2\frac{\Phi}{2}}\right).$$

The dependence between the angles $\theta_{11}$ and $\theta_{12}$, expressed by (11.19), can be represented in simpler form. To do this, using (11.19), we write the expression for the tangent of the sum of the angles $\theta_{11}$ and $\theta_{12}$. We obtain

(11.20) $$\tan(\theta_{11} + \theta_{12}) = \frac{2k_1\cot\frac{\Phi}{2}}{1 - \cot^2\frac{\Phi}{2}\left(\tan^2\frac{\Phi}{2} - 2k_1\tan\frac{\Phi}{2}\cot 2\theta_{1m}\right)}$$

or

$$\tan(\theta_{11} + \theta_{12}) = \tan 2\theta_{1m}$$

from which there also results the desired simple dependence between the angles $\theta_{11}$ and $\theta_{12}$ in the form

(11.21) $$\theta_{11} + \theta_{12} = 2\theta_{1m}$$

or

(11.22) $$\theta_{11} - \theta_{1m} = \theta_{1m} - \theta_{12} .$$

The latter relation shows graphically that two values of $\theta_1$ correspond to each value of $k_1$ satisfying condition (11.18), namely: $\theta_{11}$ and $\theta_{12}$, which determine the directions of the initial velocities, symmetric with respect to the direction of the minimum initial velocity. Two trajectories passing through the prescribed points $O_1$ and $O_2$ and corresponding to the very same value of the initial velocity correspond to these two symmetric directions. Hence, that trajectory which is located under the minimum-velocity trajectory is usually called the **curved trajectory** and that which is located above the minimum-velocity trajectory is called the **flat trajectory.*** Hence, the minimum-velocity trajectory is a sort of boundary trajectory separating the curved and flat trajectories.

This latter property of the minimum-velocity trajectory permits the expectation that introduction of the parameter

$$(11.23) \quad \Delta\theta_1 = \theta_1 - \theta_{1m}$$

in place of the parameter $\theta_1$ might simplify the original relation (11.4) for the family of identical-range trajectories. To verify this hypothesis, let us return to the relation (11.13), which is identical to (11.4), but more convenient for our purposes. Performing simple transformations with trigonometric functions, let us reduce it to the form:

$$(11.24) \quad \frac{1}{k_1} = \cot\frac{\Phi}{2}\,\frac{\cos 2(\theta_1 - \theta_{1m}) + \cos 2\theta_{1m}}{\sin 2\theta_{1m}}$$

after which it is possible to transform, without difficulty, from the parameter $\theta_1$ to the parameter $\Delta\theta_1$ and to write

$$(11.25) \quad k_1 = \frac{\sin 2\theta_{1m}}{\cos 2\Delta\theta_1 + \cos 2\theta_{1m}} \tan\frac{\Phi}{2}$$

or

$$(11.26) \quad k_1 = \frac{2k_{1m}\cos^2\theta_{1m}}{\cos 2\Delta\theta_1 + \cos 2\theta_{1m}} .$$

Using the fact that the denominator of (11.26) can be represented by means of simple trigonometric transformations in the form:

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*In this very graphic definition for elliptical trajectories, we speak only of the sections of the trajectories included between the points $O_1$ and $O_2$ and located on a sphere of radius $r_2$. To apply this definition to the parabolic and hyperbolic trajectories, it is necessary to consider the location of sections containing the infinitely distant point under the minimum-velocity trajectory.

$$\cos 2\alpha + \cos 2\beta = (\cos^2\alpha - \sin^2\beta) + (\cos^2\beta - \sin^2\alpha)$$

$$= 2(\cos^2\alpha - \sin^2\beta) = 2(\cos^2\beta - \sin^2\alpha)$$

let us rewrite (11.26) in the form:

$$(11.27)\quad k_1 = \frac{k_{1m}\cos^2\theta_{1m}}{\cos^2\theta_{1m} - \sin^2\Delta\theta_1}$$

Formula (11.27) shows that the expected simplification of the original dependence actually occurs.

Sometimes the following relations will also be useful:

$$(11.28)\quad \frac{v_1^2 - v_{1m}^2}{v_{1m}^2} = \frac{k_1 - k_{1m}}{k_{1m}} = \frac{\sin^2\Delta\theta_1}{\cos^2\theta_{1m} - \sin^2\Delta\theta_1}$$

and

$$\frac{k_1\cos^2\theta_1}{k_{1m}\cos^2\theta_{1m}} = \frac{1 - \tan\theta_{1m}\tan\Delta\theta_1}{1 + \tan\theta_{1m}\tan\Delta\theta_1}$$

which are easily obtained from (11.27).

In conclusion, let us examine how the doubled sectorial velocity $\sigma$ changes during the transition from one trajectory of the family to another To do this, we put (4.9) for $\sigma$ in the following form:

$$(11.29)\quad \sigma = r_1v_1\cos\theta_1 = \sqrt{2\pi_0r_1k_1\cos^2\theta_1}$$

It follows from (11.29) that the change in the doubled sectorial velocity from trajectory to trajectory is determined completely by the quantity $k_1\cos^2\theta_1$, since the remaining quantities remain constant for all the trajectories of the family. On the basis of (11.27) we can write

$$(11.30)\quad k_1\cos^2\theta_1 = \frac{k_{1m}\cos^2\theta_{1m}\cos^2\theta_1}{\cos^2\theta_{1m} - \sin^2\Delta\theta_1}$$

and, having made simple trigonometric transformations, we can represent the expression for $k_1\cos^2\theta_1$ in the following form:

$$(11.31) \quad k_1\cos^2\theta_1 = \frac{k_{1m}}{(1 - \tan^2\theta_{1m}) + 2\tan\theta_{1m}\tan\theta_1}$$

In (11.31), the expression $1 - \tan^2\theta_{1m}$ is always greater than zero; hence, $k_1\cos^2\theta_1$ diminishes as $\theta_1$ increases and, therefore, the doubled sectorial velocity $\sigma$ diminishes.

The expression

$$T = \frac{2S}{\sigma}$$

can be written for the flight time T, where S is the area swept over by the radius vector during the motion of the point under consideration over the trajectory of the family; $\sigma$ is the doubled sectorial velocity of the same point, which is constant for each trajectory of the family.

It is clear from geometric considerations that the area S becomes large as $\theta_1$ increases. On the other hand, we explained somewhere above that the doubled sectorial velocity $\sigma$ decreases as $\theta_1$ increases. Therefore, the conclusion can be made that the flight time over the trajectories of the family increases monotonically as $\theta_1$ grows.

We thus terminate the analysis of the properties of identical-range trajectories. Let us just examine the following important property of a minimum-velocity trajectory, which we have not touched upon. The minimum-velocity trajectory of a family of identical-range trajectories is at the same time a maximum-range trajectory for a family of identical-velocity trajectories. Starting from this property, we introduce completely identical notations for the parameters of both trajectories since the relations obtained for the parameters of one trajectory are also valid for the parameters of the other because of this property.

The identity between the above-mentioned trajectories is proved most simply if, when analyzing the family of identical-initial-velocity trajectories, the magnitude of the initial velocity is taken equal to the magnitude of the minimum initial velocity of the family of identical-range trajectories. Evidently one trajectory among the family of identical-initial-velocity trajectories will absolutely pass through the point $O_2$

prescribed for the family of identical-range trajectories. This will be the trajectory which coincides with the minimum-velocity trajectory. Let us prove that this trajectory will be the maximum-range trajectory for the family under consideration.

Actually, if this were not so, then among the family of trajectories there would have to be some whose range exceeded the range between the points $O_1$ and $O_2$. In this case there would certainly have to be one such trajectory which would pass through the point $O_2$; but this contradicts the original assumption on the minimality of the initial velocity taken for the prescribed points $O_1$ and $O_2$, since only one trajectory passing through these points exists for the minimum initial velocity. Therefore, the minimum-velocity trajectory of the family of identical-range trajectories is at the same time the maximum-range trajectory for the family of identical-initial-velocity trajectories. Hence, all the relations obtained for the parameters of one trajectory are valid for the parameters of the other as well.

# 12. FUNDAMENTAL RELATIONS BETWEEN PARAMETERS OF THE MAXIMUM-RANGE AND MINIMUM -VELOCITY TRAJECTORIES

Because of the great practical interest represented by the maximum-range trajectory of the family of identical-initial-velocity trajectories and the minimum-velocity trajectory of the family of identical-range trajectories, it is expedient to present in a separate section, the relations between the parameters of these trajectories, obtained in sections 10 and 11. This section is in the form of a summary of the formulas intended for the solution of various practical problems.

If the parameters

$$(12.1) \qquad r_1, \ r_2, \ \Phi \left( \bar{h}_1 = \frac{r_1 - r_2}{r_2} \right)$$

of the mentioned trajectories are known, then it is convenient to define their remaining parameters by means of the following formulas:

$$\tan 2\theta_{1m} = \frac{2\cot\frac{\Phi}{2}}{2 + \bar{h}_1 + \bar{h}_1 \cot^2\frac{\Phi}{2}} \tag{12.2}$$

$$k_{1m} = \tan\frac{\Phi}{2} \tan\theta_{1m} \; ; \; v_{1m} = \sqrt{\frac{2\pi_0 k_{1m}}{r_1}} \; . \tag{12.3}$$

If

$$r_1, \; r_2, \; v_{1m}, \; k_{1m} = \frac{v_{1m}^2 r_1}{2\pi_0}$$

are known, then to determine the remaining parameters it is convenient to use

$$\tan\theta_{1m} = \sqrt{\frac{k_{1m} - k_{1m}^2 (2 + \bar{h}_1)}{k_{1m} + \bar{h}_1}} \tag{12.5}$$

$$\tan\frac{\Phi}{2} = k_{1m} \cot\theta_{1m} \tag{12.6}$$

## 13. DEVIATIONS IN THE ANGULAR FLIGHT RANGE CAUSED BY INITIAL DEVIATIONS

When analyzing motions along Keplerian trajectories it is often useful to know the dependence-relating deviations in the initial-motion conditions, which we shall simply call **initial deviations** for brevity, to the deviations in the angular flight range which they cause.

This dependence is given implicitly by (7.18), which represents $\Phi$ as a function of the quantities $v_1$, $\theta_1$, $r_1$ :

$$\Phi = \Phi(v_1, \; \theta_1, \; r_1) \tag{13.1}$$

Explicitly, it is usually represented by the Taylor formula*

$$(13.2) \quad \delta\Phi = \sum_{k=1}^{n} \delta^k\Phi + R_{n+1}$$

where

$$(13.3) \quad \delta^k\Phi = \frac{1}{k!}\left(\delta v_1 \frac{\partial}{\partial v_1} + \delta\theta_1 \frac{\partial}{\partial \theta_1} + \delta r_1 \frac{\partial}{\partial r_1}\right)^k \Phi \ ;$$

$R_{n+1}$ is the remainder term of the formula; $\delta v_1$, $\delta\theta_1$, $\delta r_1$ are initial deviations in $v$, $\theta$ and $r$ ; $\delta\Phi$ is the deviation in the angular range due to the initial deviations $\delta v_1$, $\delta\theta_1$, $\delta r_1$.

Considering the initial deviations $\delta v_1$, $\delta\theta_1$, $\delta r_1$ to be sufficiently small, one usually is limited to an approximate expression for the desired dependence; terms higher than first and second order infinitesimals are neglected in the Taylor formula.

1. Expression for $\delta^1\Phi$. The expression for $\delta^1\Phi$ can be written in the following form:**

$$(13.4) \quad \delta^1\Phi = \bar{\Phi}_{v_1} \delta\bar{v}_1 + \bar{\Phi}_{\theta_1} \delta\bar{\theta}_1 + \bar{\Phi}_{r_1} \delta\bar{r}_1$$

where

$$(13.5) \quad \bar{\Phi}_{v_1} = \frac{\partial\Phi}{\partial v_1} v_1 \ ; \ \bar{\Phi}_{\theta_1} = \frac{\partial\Phi}{\partial\theta_1} \ ; \ \bar{\Phi}_{r_1} = \frac{\partial\Phi}{\partial r_1} r_1$$

$$(13.6) \quad \delta\bar{v}_1 = \frac{\delta v_1}{v_1} \ ; \ \delta\bar{\theta}_1 = \delta\theta_1 \ ; \ \delta\bar{r}_1 = \frac{\delta r_1}{r_1}$$

where the angles are measured in radians.

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*Here and everywhere henceforth the symbolic form of writing the Taylor formula, which is convenient for its simplicity and clarity, is used.

**As before, we shall denote a non-dimensional quantity by a bar above the letter. The bar will also be set over the angular quantities for symmetry in the notation.

To determine the quantities in (13.5) it is necessary to find the partial derivatives of $\Phi$ with respect to the parameters $v_1$, $\theta_1$, $r_1$.

One of these derivatives was determined in section 10, namely, the derivative $\partial \Phi \; \partial \theta_1$, by differentiation of (7.16). Let us find the remaining derivatives in the same manner. Let q, conditionally, denote the variable with respect to which we shall carry out the differentiation. Then we can write

$$(13.7) \quad - \frac{\chi_1 \tan \frac{\Phi}{2} - \tan \theta_1}{\cos^2 \frac{\Phi}{2}} \frac{\partial \Phi}{\partial q} = \tan^2 \frac{\Phi}{2} \cdot \frac{\partial \chi_1}{\partial q} - 2 \tan \frac{\Phi}{2} \frac{\partial \tan \theta_1}{\partial q} - \frac{\partial \bar{h}_1}{\partial q} .$$

In order to obtain the desired derivatives by using the above formula, let us write the expression for the partial derivatives on the right side. We shall here assume q to be not only the parameters $v_1$, $\theta_1$, $r_1$ but also the parameters $\pi_0$ and $r_2$, since the deviations in range caused by deviations in the parameters $\pi_0$ and $r_2$ are also of considerable interest and can be obtained in passing together with the desired deviations. Thus, putting

$$(13.8) \quad q = v_1, \theta_1, r_1, \pi_0, r_2$$

we obtain:

$$(13.9) \quad \frac{\partial \bar{h}_1}{\partial v_1} = \frac{\partial \bar{h}_1}{\partial \theta_1} = \frac{\partial \bar{h}_1}{\partial \pi_0} = 0 ; \quad \frac{\partial \bar{h}_1}{\partial r_1} = \frac{1}{r_2} ; \quad \frac{\partial \bar{h}_1}{\partial r_2} = - \frac{r_1}{r_2^2}$$

$$(13.10) \quad \frac{\partial \tan \theta_1}{\partial v_1} = \frac{\partial \tan \theta_1}{\partial r_1} = \frac{\partial \tan \theta_1}{\partial \pi_0} = \frac{\partial \tan \theta_1}{\partial r_2} = 0 ; \quad \frac{\partial \tan \theta_1}{\partial \theta_1} = \frac{1}{\cos^2 \theta_1}$$

$$(13.11) \quad \left\{ \begin{aligned} & \frac{\partial \chi_1}{\partial v_1} = - \frac{2}{k_1 \cos^2 \theta_1} \cdot \frac{1}{v_1} ; \quad \frac{\partial \chi_1}{\partial \pi_0} = \frac{1}{k_1 \cos^2 \theta_1} \cdot \frac{1}{\pi_0} ; \quad \frac{\partial \chi_1}{\partial \theta_1} = \frac{2 \tan \theta_1}{k_1 \cos^2 \theta_1} \\ & \frac{\partial \chi_1}{\partial r_2} = \frac{r_1}{r_2^2} ; \quad \frac{\partial \chi_1}{\partial r_1} = - \left( \frac{1}{r_2} + \frac{1}{r_1 k_1 \cos^2 \theta_1} \right) . \end{aligned} \right.$$

Substituting (13.9), (13.10) and (13.11) into (13.7) and using (8.8), let us write the desired expressions as follows:

$$
(13.12)\begin{cases}
\dfrac{\partial\Phi}{\partial v_1} = \varphi_1 \dfrac{2\sin^2\frac{\Phi}{2}}{k_1\cos^2\theta_1}\,\dfrac{1}{v_1} \\[2ex]
\dfrac{\partial\Phi}{\partial\theta_1} = \varphi_1 \dfrac{\sin\Phi}{k_1\cos^2\theta_1}\left(k_1 - \tan\theta_1\tan\dfrac{\Phi}{2}\right) = v_1\left(k_1\cot\dfrac{\Phi}{2} - \tan\theta_1\right)\dfrac{\partial\Phi}{\partial v_1} \\[2ex]
\dfrac{\partial\Phi}{\partial r_1} = \varphi_1\left(\dfrac{1}{r_2} + \dfrac{\sin^2\frac{\Phi}{2}}{k_1 r_1\cos^2\theta_1}\right) = \dfrac{v_1}{2r_1}\cdot\dfrac{\partial\Phi}{\partial v_1} + \dfrac{\varphi_1}{r_2} \\[2ex]
\dfrac{\partial\Phi}{\partial\pi_0} = -\varphi_1\dfrac{\sin^2\frac{\Phi}{2}}{k_1\cos^2\theta_1}\cdot\dfrac{1}{\pi_0} = -\dfrac{v_1}{2\pi_0}\cdot\dfrac{\partial\Phi}{\partial v_1}\,;\quad \dfrac{\partial\Phi}{\partial r_2} = -\varphi_1\dfrac{r_1}{r_2^2} \\[2ex]
\varphi_1 = \dfrac{1}{\tan\theta_1 + \bar{h}_1\cot\frac{\Phi}{2}} \qquad \bar{h}_1 = \dfrac{r_1 - r_2}{r_2}
\end{cases}
$$

As a result, the expression for the deviation of the angular range $\delta^1\Phi$ caused by deviations in the parameters (13.8) can be written as follows:

$$
(13.13)\quad \delta^1\Phi = \bar{\Phi}_{v_1}\,\delta\bar{v}_1 + \bar{\Phi}_{\theta_1}\,\delta\bar{\theta}_1 + \bar{\Phi}_{r_1}\,\delta\bar{r}_1 + \bar{\Phi}_{\pi_0}\,\delta\bar{\pi}_0 + \bar{\Phi}_{r_2}\,\delta\bar{r}_2
$$

where

$$(13.14)\quad \begin{cases} \bar{\Phi}_{v_1} = \dfrac{\partial \Phi}{\partial v_1}\, v_1 = \varphi_1 \dfrac{2 \sin^2 \frac{\Phi}{2}}{k_1 \cos^2 \theta_1} \\[2ex] \bar{\Phi}_{\theta_1} = \dfrac{\partial \Phi}{\partial \theta_1} = \left( k_1 \cot \dfrac{\Phi}{2} - \tan \theta_1 \right) \bar{\Phi}_{v_1}\,. \\[2ex] \bar{\Phi}_{r_1} = \dfrac{\partial \Phi}{\partial r_1}\, r_1 = \dfrac{1}{2}\, \bar{\Phi}_{v_1} + \varphi_1 \dfrac{r_1}{r_2} \\[2ex] \bar{\Phi}_{\pi_0} = \dfrac{\partial \Phi}{\partial \pi_0}\, \pi_0 = - \dfrac{1}{2}\, \bar{\Phi}_{v_1} \;; \quad \bar{\Phi}_{r_2} = \dfrac{\partial \Phi}{\partial r_2}\, r_2 = - \varphi_1 \dfrac{r_1}{r_2} \\[2ex] \varphi_1 = \dfrac{1}{\tan \theta_1 + \bar{h}_1 \cot \frac{\Phi}{2}} \;; \quad \bar{h}_1 = \dfrac{r_1 - r_2}{r_2} \;; \quad k_1 = \dfrac{v_1^2 r_1}{2\pi_0} \end{cases}$$

$$(13.15)\quad \delta \bar{v}_1 = \frac{\delta v_1}{v_1} \;; \quad \delta \bar{\theta}_1 = \delta \theta_1 \;; \quad \delta \bar{r}_1 = \frac{\delta r_1}{r_1} \;; \quad \delta \bar{r}_2 = \frac{\delta r_2}{r_2} \;; \quad \delta \bar{\pi}_0 = \frac{\delta \pi_0}{\pi_0}\,.$$

2. **Expression for** $\delta^2\Phi$. The expression for $\delta^2\Phi$ can be written as follows:

$$(13.16)\quad \delta^2\Phi = \bar{\Phi}_{v_1 v_1} \delta \bar{v}_1^2 + \bar{\Phi}_{\theta_1 \theta_1} \delta \bar{\theta}_1^2 + \bar{\Phi}_{r_1 r_1} \delta \bar{r}_1^2 + \bar{\Phi}_{v_1 \theta_1} \delta \bar{v}_1 \delta \theta_1 + \bar{\Phi}_{v_1 r_1} \delta \bar{v}_1 \delta r_1 + \bar{\Phi}_{\theta_1 r_1} \delta \bar{\theta}_1 \delta r_1$$

where

$$(13.17)\quad \begin{cases} \bar{\Phi}_{v_1 v_1} = \dfrac{1}{2}\, v_1^2 \dfrac{\partial^2 \Phi}{\partial v_1^2} \;; \quad \bar{\Phi}_{\theta_1 \theta_1} = \dfrac{1}{2} \dfrac{\partial^2 \Phi}{\partial \theta_1^2} \;; \quad \bar{\Phi}_{r_1 r_1} = \dfrac{1}{2}\, r_1^2 \dfrac{\partial^2 \Phi}{\partial r_1^2} \\[2ex] \bar{\Phi}_{v_1 \theta_1} = v_1 \dfrac{\partial^2 \Phi}{\partial v_1 \partial \theta_1} \;; \quad \bar{\Phi}_{v_1 r_1} = v_1 r_1 \dfrac{\partial^2 \Phi}{\partial v_1 \partial r_1} \;; \quad \bar{\Phi}_{\theta_1 r_1} = r_1 \dfrac{\partial^2 \Phi}{\partial \theta_1 \partial r_1} \end{cases}$$

and $\delta \bar{v}_1$, $\delta \bar{\theta}_1$, $\delta \bar{r}_1$ are defined, as before, by (13.6).

To determine the quantities in (13.17) it is necessary to find the second partial derivatives of $\Phi$ with respect to the parameters $r_1$, $v_1$, $\theta_1$. This is done most simply by differentiating the first derivatives (13.12) obtained above. Hence, as before, we determine the second partial derivatives with respect to all the parameters of (13.8) in passing.

Returning to (13.12), we note that the majority of the first derivatives are expressed in terms of the derivative $\partial\Phi/\partial v_1$.

Hence, by determining all the partial derivatives of this derivative, we facilitate the determination of the remaining second derivatives. By denoting by q the variable with respect to which the differentiation is carried out conditionally, we can write, starting from (13.12)

$$(13.18)\quad \frac{\partial^2\Phi}{\partial v_1 \partial q} = \frac{\partial\Phi}{\partial v_1}\left[\frac{1}{\varphi_1}\frac{\partial\varphi_1}{\partial q} + \frac{1}{\sin^2\frac{\Phi}{2}}\frac{\partial \sin^2\frac{\Phi}{2}}{\partial q} + k_1 v_1 \frac{\partial}{\partial q}\left(\frac{1}{k_1 v_1}\right)\right.$$
$$\left. + \cos^2\theta_1 \frac{\partial}{\partial q}\left(\frac{1}{\cos^2\theta_1}\right)\right]$$

Substituting the expression for $\partial\varphi_1/\partial q$ here, which is easily represented in the form

$$(13.19)\quad \frac{\partial\varphi_1}{\partial q} = \varphi_1^2\left(\frac{\bar{h}_1}{2\sin^2\frac{\Phi}{2}}\frac{\partial\Phi}{\partial q} - \cot\frac{\Phi}{2}\frac{\partial\bar{h}_1}{\partial q} - \frac{\partial \tan\theta_1}{\partial q}\right)$$

we obtain

$$(13.20)\quad \frac{\partial^2\Phi}{\partial v_1 \partial q} = \frac{\varphi_1 \bar{h}_1}{2\sin^2\frac{\Phi}{2}} + \cot\frac{\Phi}{2}\frac{\partial\Phi}{\partial v_1}\frac{\partial\Phi}{\partial q}$$
$$+ \left[(\sin 2\theta_1 - \varphi_1)\frac{\partial \tan\theta_1}{\partial q} - \varphi_1 \cot\frac{\Phi}{2}\frac{\partial\bar{h}_1}{\partial q}\right.$$
$$\left. - \frac{1}{k_1 v_1}\frac{\partial(k_1 v_1)}{\partial q}\right]\frac{\partial\Phi}{\partial v_1}.$$

Using formula (13.20), we can write the following derivatives without special difficulties:

$$(13.21)\quad \frac{\partial^2\Phi}{\partial v_1^2} = \left(\cot\frac{\Phi}{2} + \frac{\varphi_1\bar{h}_1}{2\sin^2\frac{\Phi}{2}}\right)\left(\frac{\partial\Phi}{\partial v_1}\right)^2 - \frac{3}{v_1}\frac{\partial\Phi}{\partial v_1}$$

$$(13.22)\quad \frac{\partial^2\Phi}{\partial v_1\partial\theta_1} = \left(\cot\frac{\Phi}{2} + \frac{\varphi_1\bar{h}_1}{2\sin^2\frac{\Phi}{2}}\right)\frac{\partial\Phi}{\partial v_1}\frac{\partial\Phi}{\partial\theta_1} + \left(2\tan\theta_1 - \frac{\varphi_1}{\cos^2\theta_1}\right)\frac{\partial\Phi}{\partial v_1}$$

$$(13.23)\quad \frac{\partial^2\Phi}{\partial v_1\partial r_1} = \left(\cot\frac{\Phi}{2} + \frac{\varphi_1\bar{h}_1}{2\sin^2\frac{\Phi}{2}}\right)\frac{\partial\Phi}{\partial v_1}\frac{\partial\Phi}{\partial r_1} - \left(\frac{1}{r_1} + \frac{\varphi_1}{r_2}\cot\frac{\Phi}{2}\right)\frac{\partial\Phi}{\partial v_1}$$

$$(13.24)\quad \frac{\partial^2\Phi}{\partial v_1\partial\pi_0} = \left(\cot\frac{\Phi}{2} + \frac{\varphi_1\bar{h}_1}{2\sin^2\frac{\Phi}{2}}\right)\frac{\partial\Phi}{\partial v_1}\frac{\partial\Phi}{\partial\pi_0} + \frac{1}{\pi_0}\frac{\partial\Phi}{\partial v_1}$$

$$(13.25)\quad \frac{\partial^2\Phi}{\partial v_1\partial r_2} = \left(\cot\frac{\Phi}{2} + \frac{\varphi_1\bar{h}_1}{2\sin^2\frac{\Phi}{2}}\right)\frac{\partial\Phi}{\partial v_1}\frac{\partial\Phi}{\partial r_2} + \frac{r_1}{r_2}\varphi_1\cot\frac{\Phi}{2}\frac{\partial\Phi}{\partial v_1}$$

$$= \frac{\varphi_1\bar{h}_1}{2\sin^2\frac{\Phi}{2}}\frac{\partial\Phi}{\partial v_1}\frac{\partial\Phi}{\partial r_2}$$

Now it is convenient to return to the derivative $\partial\Phi/\partial r_2$ and to differentiate it with respect to all the parameters of (13.8) except $v_1$. Here, as in the preceding case, it is convenient at the beginning to write the general expression for the derivative of $\partial\Phi/\partial r_2$ with respect to q. Using (13.19) this expression can be represented as

$$(13.26)\quad \frac{\partial^2\Phi}{\partial r_2\partial q} = -\frac{r_1}{r_2^2}\frac{\partial\varphi_1}{\partial q} - \frac{\varphi_1}{r_2^2}\frac{\partial r_1}{\partial q} + \frac{2\varphi_1 r_1}{r_2^3}\frac{\partial r_2}{\partial q}$$

$$= \frac{\partial\Phi}{\partial r_2}\left[\varphi_1\left(\frac{\bar{h}_1}{2\sin^2\frac{\Phi}{2}}\frac{\partial\Phi}{\partial q} - \cot\frac{\Phi}{2}\frac{\partial\bar{h}_1}{\partial q} - \frac{\partial\tan\theta_1}{\partial q}\right) + \frac{1}{r_1}\frac{\partial r_1}{\partial q} - \frac{2}{r_2}\frac{\partial r_2}{\partial q}\right].$$

Using (13.26), the following derivatives can be written without special difficulty:

$$(13.27)\quad \frac{\partial^2\Phi}{\partial r_2 \partial\theta_1} = \frac{\varphi_1 \bar{h}_1}{2\sin^2\frac{\Phi}{2}} \frac{\partial\Phi}{\partial r_2} \frac{\partial\Phi}{\partial\theta_1} - \frac{\varphi_1}{\cos^2\theta_1} \frac{\partial\Phi}{\partial r_2}$$

$$= \left(\cot\frac{\Phi}{2} + \frac{\varphi_1 \bar{h}_1}{2\sin^2\frac{\Phi}{2}}\right) \frac{\partial\Phi}{\partial r_2} \frac{\partial\Phi}{\partial\theta_1}$$

$$- \left(\frac{\varphi_1}{\cos^2\theta_1} + \cot\frac{\Phi}{2} \frac{\partial\Phi}{\partial\theta_1}\right) \frac{\partial\Phi}{\partial r_2}$$

$$(13.28)\quad \frac{\partial^2\Phi}{\partial r_2 \partial r_1} = \frac{\varphi_1 \bar{h}_1}{2\sin^2\frac{\Phi}{2}} \frac{\partial\Phi}{\partial r_2} \frac{\partial\Phi}{\partial r_1} + \left(\frac{1}{r_1} - \frac{\varphi_1}{r_2}\cot\frac{\Phi}{2}\right) \frac{\partial\Phi}{\partial r_2}$$

$$= \left(\cot\frac{\Phi}{2} + \frac{\varphi_1 \bar{h}_1}{2\sin^2\frac{\Phi}{2}}\right) \frac{\partial\Phi}{\partial r_2} \frac{\partial\Phi}{\partial r_1}$$

$$+ \left(\frac{1}{r_1} - \frac{2\varphi_1}{r_2}\cot\frac{\Phi}{2} - \frac{v_1}{2r_1}\cot\frac{\Phi}{2} \frac{\partial\Phi}{\partial\theta_1}\right) \frac{\partial\Phi}{\partial r_2}$$

$$(13.29)\quad \frac{\partial^2\Phi}{\partial r_2 \partial\pi_0} = \frac{\varphi_1 \bar{h}_1}{2\sin^2\frac{\Phi}{2}} \frac{\partial\Phi}{\partial r_2} \frac{\partial\Phi}{\partial\pi_0}$$

$$= \left(\cot\frac{\Phi}{2} + \frac{\varphi_1 \bar{h}_1}{2\sin^2\frac{\Phi}{2}}\right) \frac{\partial\Phi}{\partial r_2} \frac{\partial\Phi}{\partial\pi_0} - \cot\frac{\Phi}{2} \frac{\partial\Phi}{\partial r_2} \frac{\partial\Phi}{\partial\pi_0}$$

$$(13.30)\quad \frac{\partial^2\Phi}{\partial r_2^2} = \frac{\varphi_1 \bar{h}_1}{2\sin^2\frac{\Phi}{2}} \left(\frac{\partial\Phi}{\partial r_2}\right)^2 + \left(\frac{\varphi_1 r_1}{r_2^2}\cot\frac{\Phi}{2} - \frac{2}{r_2}\right) \frac{\partial\Phi}{\partial r_2}$$

$$= \left(\cot\frac{\Phi}{2} + \frac{\varphi_1 \bar{h}_1}{2\sin^2\frac{\Phi}{2}}\right) \left(\frac{\partial\Phi}{\partial r_2}\right)^2 - \left(\frac{2}{r_2} + 2\cot\frac{\Phi}{2} \frac{\partial\Phi}{\partial r_2}\right) \frac{\partial\Phi}{\partial r_2}$$

Furthermore, let us differentiate the derivative $\partial\Phi/\partial\pi_0$ with respect to all the parameters of (13.8) except $v_1$ and $r_1$. As a result we obtain

$$\begin{aligned}\frac{\partial^2\Phi}{\partial\pi_0\partial\theta_1} &= -\frac{v_1}{2\pi_0}\frac{\partial^2\Phi}{\partial v_1\partial\theta_1} \\ &= \left(\cot\frac{\Phi}{2} + \frac{\varphi_1\bar{h}_1}{2\sin^2\frac{\Phi}{2}}\right)\frac{\partial\Phi}{\partial\pi_0}\frac{\partial\Phi}{\partial\theta_1} + \left(2\tan\theta_1 - \frac{\varphi_1}{\cos^2\theta_1}\right)\frac{\partial\Phi}{\partial\pi_0}\end{aligned} \tag{13.31}$$

$$\begin{aligned}\frac{\partial^2\Phi}{\partial\pi_0\partial r_1} &= -\frac{v_1}{2\pi_0}\frac{\partial^2\Phi}{\partial v_1\partial r_1} \\ &= \left(\cot\frac{\Phi}{2} + \frac{\varphi_1\bar{h}_1}{2\sin^2\frac{\Phi}{2}}\right)\frac{\partial\Phi}{\partial\pi_0}\frac{\partial\Phi}{\partial r_1} - \left(\frac{1}{r_1} + \frac{\varphi_1}{r_2}\cot\frac{\Phi}{2}\right)\frac{\partial\Phi}{\partial\pi_0}\end{aligned} \tag{13.32}$$

$$\frac{\partial^2\Phi}{\partial\pi_0^2} = -\frac{v_1}{2\pi_0}\frac{\partial^2\Phi}{\partial v_1\partial\pi_0} + \frac{v_1}{2\pi_0^2}\frac{\partial\Phi}{\partial v_1} = \left(\cot\frac{\Phi}{2} + \frac{\varphi_1\bar{h}_1}{2\sin^2\frac{\Phi}{2}}\right)\left(\frac{\partial\Phi}{\partial\pi_0}\right)^2 \tag{13.33}$$

After this, let us write the expression for the derivative $\partial\Phi/\partial r_1$ in the form:

$$\frac{\partial\Phi}{\partial r_1} = \frac{v_1}{2r_1}\frac{\partial\Phi}{\partial v_1} - \frac{r_2}{r_1}\frac{\partial\Phi}{\partial r_2} \tag{13.34}$$

and let us differentiate it with respect to $\theta_1$ and $r_1$. As a result we will have

$$\begin{aligned}\frac{\partial^2\Phi}{\partial r_1\partial\theta_1} &= \frac{v_1}{2r_1}\frac{\partial^2\Phi}{\partial v_1\partial\theta_1} - \frac{r_2}{r_1}\frac{\partial^2\Phi}{\partial r_2\partial\theta_1} \\ &= \left(\cot\frac{\Phi}{2} + \frac{\varphi_1\bar{h}_1}{2\sin^2\frac{\Phi}{2}}\right)\frac{\partial\Phi}{\partial r_1}\frac{\partial\Phi}{\partial\theta_1} - \frac{\varphi_1}{\cos^2\theta_1}\frac{\partial\Phi}{\partial r_1} \\ &\quad + \frac{v_1}{r_1}\tan\theta_1\frac{\partial\Phi}{\partial v_1} - \frac{\varphi_1}{r_2}\cot\frac{\Phi}{2}\frac{\partial\Phi}{\partial\theta_1}\end{aligned} \tag{13.35a}$$

$$(13.35b)\quad \frac{\partial^2\Phi}{\partial r_1^2} = \frac{v_1}{2r_1}\,\frac{\partial^2\Phi}{\partial v_1\partial r_1} - \frac{r_2}{r_1}\,\frac{\partial^2\Phi}{\partial r_2\partial r_1} - \frac{1}{r_1}\,\frac{\partial\Phi}{\partial r_1}$$

$$= \left(\cot\frac{\Phi}{2} + \frac{\varphi_1\bar{h}_1}{2\sin^2\frac{\Phi}{2}}\right)\frac{\partial\Phi}{\partial r_1}\,\frac{\partial\Phi}{\partial r_1} - \left(\frac{2}{r_1} + \frac{2\varphi_1}{r_2}\cot\frac{\Phi}{2}\right)\frac{\partial\Phi}{\partial r_1} + \frac{2\varphi_1}{r_1r_2}\,.$$

We determine the last remaining derivative by differentiating $\partial\Phi/\partial\theta_1$ with respect to $\theta_1$. After differentiation we obtain

$$(13.36)\quad \frac{\partial^2\Phi}{\partial\theta_1^2} = v_1\left(k_1\cot\frac{\Phi}{2} - \tan\theta_1\right)\frac{\partial^2\Phi}{\partial v_1\partial\theta_1} - \frac{v_1k_1}{2\sin^2\frac{\Phi}{2}}\,\frac{\partial\Phi}{\partial\theta_1}\,\frac{\partial\Phi}{\partial v_1}$$

$$- \frac{v_1}{\cos^2\theta_1}\,\frac{\partial\Phi}{\partial v_1}\,.$$

Substituting (13.22) here, we rewrite the expression for the desired derivative as:

$$(13.37)\quad \frac{\partial^2\Phi}{\partial\theta_1^2} = \left(\cot\frac{\Phi}{2} + \frac{\varphi_1\bar{h}_1}{2\sin^2\frac{\Phi}{2}}\right)\frac{\partial\Phi}{\partial\theta_1}\left\{v_1\left(k_1\cot\frac{\Phi}{2} - \tan\theta_1\right)\frac{\partial\Phi}{\partial v_1}\right\}$$

$$+ \left(2\tan\theta_1 - \frac{\varphi_1}{\cos^2\theta_1}\right)\left\{v_1\left(k_1\cot\frac{\Phi}{2} - \tan\theta_1\right)\frac{\partial\Phi}{\partial v_1}\right\}$$

$$- \left(\frac{v_1k_1}{2\sin^2\frac{\Phi}{2}}\,\frac{\partial\Phi}{\partial v_1}\right)\frac{\partial\Phi}{\partial\theta_1} - \frac{v_1}{\cos^2\theta_1}\,\frac{\partial\Phi}{\partial v_1}$$

Replacing the derivative $\partial\Phi/\partial v_1$ in the parenthesis by its expression and the expression in the braces by the derivative $\partial\Phi/\partial\theta_1$ and making other obvious transformations, let us write the expression for the last derivative in the following form:

$$(13.38)\quad \frac{\partial^2\Phi}{\partial\theta_1^2} = \left(\cot\frac{\Phi}{2} + \frac{\varphi_1\bar{h}_1}{2\sin^2\frac{\Phi}{2}}\right)\left(\frac{\partial\Phi}{\partial\theta_1}\right)^2 + \left(2\tan\theta_1 - \frac{2\varphi_1}{\cos^2\theta_1}\right)\frac{\partial\Phi}{\partial\theta_1} - \frac{v_1}{\cos^2\theta_1}\frac{\partial\Phi}{\partial v_1}\ .$$

Hence, an expression can be written for the deviation $\delta^2\Phi$ in the angular range caused by deviations in all the parameters in (13.8), as follows:

$$(13.39)\quad \delta^2\Phi = \bar{\Phi}_{v_1v_1}\delta\bar{v}_1^2 + \bar{\Phi}_{\theta_1\theta_1}\delta\bar{\theta}_1^2 + \bar{\Phi}_{r_1r_1}\delta\bar{r}_1^2 + \bar{\Phi}_{v_1\theta_1}\delta\bar{v}_1\delta\bar{\theta}_1$$

$$+ \bar{\Phi}_{v_1r_1}\delta\bar{v}_1\delta\bar{r}_1 + \bar{\Phi}_{\theta_1r_1}\delta\bar{\theta}_1\delta\bar{r}_1 + \bar{\Phi}_{\pi_0\pi_0}\delta\bar{\pi}_0^2$$

$$+ \bar{\Phi}_{r_2r_2}\delta\bar{r}_2^2 + \bar{\Phi}_{\pi_0r_2}\delta\bar{\pi}_0\delta\bar{r}_2 + \bar{\Phi}_{\pi_0v_1}\delta\bar{\pi}_0\delta\bar{v}_1$$

$$+ \bar{\Phi}_{r_2v_1}\delta\bar{r}_2\delta\bar{v}_1 + \bar{\Phi}_{\pi_0\theta_1}\delta\bar{\pi}_0\delta\bar{\theta}_1$$

$$+ \bar{\Phi}_{r_2\theta_1}\delta\bar{r}_2\delta\bar{\theta}_1 + \bar{\Phi}_{\pi_0r_1}\delta\bar{\pi}_0\delta\bar{r}_1 + \bar{\Phi}_{r_2r_1}\delta\bar{r}_2\delta\bar{r}_1$$

where

$$(13.40)\quad \bar{\Phi}_{v_1v_1} = \frac{1}{2}\frac{\partial^2\Phi}{\partial v_1^2}v_1^2 = \frac{1}{2}\varphi_2\bar{\Phi}_{v_1}\bar{\Phi}_{v_1} - \frac{3}{2}\bar{\Phi}_{v_1}$$

$$(13.41)\quad \bar{\Phi}_{\theta_1\theta_1} = \frac{1}{2}\frac{\partial^2\Phi}{\partial\theta_1^2} = \frac{1}{2}\varphi_2\bar{\Phi}_{\theta_1}\bar{\Phi}_{\theta_1} + \left(\tan\theta_1 - \frac{\varphi_1}{\cos^2\theta_1}\right)\bar{\Phi}_{\theta_1} - \frac{1}{2\cos^2\theta_1}\bar{\Phi}_{v_1}$$

$$\bar{\Phi}_{r_1 r_1} = \frac{1}{2} r_1^2 \frac{\partial^2 \Phi}{\partial r_1^2} = \frac{1}{2} \varphi_2 \bar{\Phi}_{r_1} \bar{\Phi}_{r_1} + \varphi_1 (1 + \bar{h}_1) - \left[1 + \varphi_1 (1 + \bar{h}_1) \cot \frac{\Phi}{2}\right] \bar{\Phi}_{r_1} \tag{13.42}$$

$$\bar{\Phi}_{v_1 \theta_1} = v_1 \frac{\partial^2 \Phi}{\partial v_1 \partial \theta_1} = \varphi_2 \bar{\Phi}_{v_1} \bar{\Phi}_{\theta_1} + \left(2 \tan \theta_1 - \frac{\varphi_1}{\cos^2 \theta_1}\right) \bar{\Phi}_{v_1} \tag{13.43}$$

$$\bar{\Phi}_{v_1 r_1} = v_1 r_1 \frac{\partial^2 \Phi}{\partial v_1 \partial r_1} = \varphi_2 \bar{\Phi}_{v_1} \bar{\Phi}_{r_1} - \left[1 + \varphi_1 (1 + \bar{h}_1) \cot \frac{\Phi}{2}\right] \bar{\Phi}_{v_1} \tag{13.44}$$

$$\bar{\Phi}_{\theta_1 r_1} = r_1 \frac{\partial^2 \Phi}{\partial \theta_1 \partial r_1} = \varphi_2 \bar{\Phi}_{\theta_1} \bar{\Phi}_{r_1} - \frac{\varphi_1}{\cos^2 \theta_1} \bar{\Phi}_{r_1} - \varphi_1 (1 + \bar{h}_1) \cot \frac{\Phi}{2} \bar{\Phi}_{\theta_1} + \tan \theta_1 \bar{\Phi}_{v_1} \tag{13.45}$$

$$\bar{\Phi}_{\pi_0 \pi_0} = \frac{1}{2} \pi_0^2 \frac{\partial^2 \Phi}{\partial \pi_0^2} = \frac{1}{2} \varphi_2 \bar{\Phi}_{\pi_0} \bar{\Phi}_{\pi_0} \tag{13.46}$$

$$\bar{\Phi}_{r_2 r_2} = \frac{1}{2} r_2^2 \frac{\partial^2 \Phi}{\partial r_2^2} = \frac{1}{2} \varphi_2 \bar{\Phi}_{r_2} \bar{\Phi}_{r_2} - \left(1 + \cot \frac{\Phi}{2} \bar{\Phi}_{r_2}\right) \bar{\Phi}_{r_2} \tag{13.47}$$

$$\bar{\Phi}_{\pi_0 r_2} = \pi_0 r_2 \frac{\partial^2 \Phi}{\partial \pi_0 \partial r_2} = \varphi_2 \bar{\Phi}_{\pi_0} \bar{\Phi}_{r_2} - \cot \frac{\Phi}{2} \bar{\Phi}_{\pi_0} \bar{\Phi}_{r_2} = \frac{\varphi_1 \bar{h}_1}{2 \sin^2 \frac{\Phi}{2}} \bar{\Phi}_{\pi_0} \bar{\Phi}_{r_2} \tag{13.48}$$

$$\bar{\Phi}_{\pi_0 v_1} = \pi_0 v_1 \frac{\partial^2 \Phi}{\partial \pi_0 \partial v_1} = \varphi \bar{\Phi}_{\pi_0} \bar{\Phi}_{v_1} + \bar{\Phi}_{v_1} \tag{13.49}$$

$$(13.50)\quad \bar{\Phi}_{r_2 v_1} = r_2 v_1 \frac{\partial^2 \Phi}{\partial r_2 \partial v_1} = \varphi_2 \bar{\Phi}_{r_2} \bar{\Phi}_{v_1} + \varphi_1 (1 + \bar{h}_1) \cot \frac{\Phi}{2} \bar{\Phi}_{v_1}$$

$$= \frac{\varphi_1 \bar{h}_1}{2 \sin^2 \frac{\Phi}{2}} \bar{\Phi}_{r_2} \bar{\Phi}_{v_1}$$

$$(13.51)\quad \bar{\Phi}_{\pi_0 \theta_1} = \pi_0 \frac{\partial^2 \Phi}{\partial \pi_0 \partial \theta_1} = \varphi_2 \bar{\Phi}_{\pi_0} \bar{\Phi}_{\theta_1} + \left(2 \tan \theta_1 - \frac{\varphi_1}{\cos^2 \theta_1}\right) \bar{\Phi}_{\pi_0}$$

$$(13.52)\quad \bar{\Phi}_{r_2 \theta_1} = r_2 \frac{\partial^2 \Phi}{\partial r_2 \partial \theta_1} = \varphi_2 \bar{\Phi}_{r_2} \bar{\Phi}_{\theta_1} - \left(\frac{\varphi_1}{\cos^2 \theta_1} + \cot \frac{\Phi}{2} \frac{\partial \Phi}{\partial \theta_1}\right) \bar{\Phi}_{r_2}$$

$$(13.53)\quad \bar{\Phi}_{\pi_0 r_1} = \pi_0 r_1 \frac{\partial^2 \Phi}{\partial \pi_0 \partial r_1} = \varphi_2 \bar{\Phi}_{\pi_0} \bar{\Phi}_{r_1} - \left[1 + \varphi_1 (1 + \bar{h}_1) \cot \frac{\Phi}{2}\right] \bar{\Phi}_{\pi_0}$$

$$(13.54)\quad \bar{\Phi}_{r_2 r_1} = r_2 r_1 \frac{\partial^2 \Phi}{\partial r_2 \partial r_1}$$

$$= \varphi_2 \bar{\Phi}_{r_2} \bar{\Phi}_{r_1} + \left[1 - 2 \varphi_1 (1 + \bar{h}_1) \cot \frac{\Phi}{2} - \frac{1}{2} \cot \frac{\Phi}{2} \bar{\Phi}_{v_1}\right] \bar{\Phi}_{r_2}$$

$$(13.55)\quad \varphi_2 = \cot \frac{\Phi}{2} + \frac{\varphi_1 \bar{h}_1}{2 \sin^2 \frac{\Phi}{2}}; \qquad \varphi_1 = \frac{1}{\tan \theta_1 + \bar{h}_1 \cot \frac{\Phi}{2}}$$

and $\delta \bar{v}_1$, $\delta \bar{\theta}_1$, $\delta \bar{r}_1$, $\delta \bar{\pi}_0$, $\delta \bar{r}_2$ are defined by (13.15).

In certain cases the deviations in angular range caused by the initial deflections are conveniently represented in the form

$$(13.56)\quad \delta^2 \Phi = \bar{\bar{\Phi}}_{\Phi\Phi} \delta^1 \Phi^2 + \bar{\bar{\Phi}}_{\Phi \theta_1} \delta^1 \Phi \delta \bar{\theta}_1 + \bar{\bar{\Phi}}_{\Phi r_1} \delta^1 \Phi \delta \bar{r}_1$$

$$+ \bar{\bar{\Phi}}_{\theta_1 \theta_1} \delta \bar{\theta}_1^2 + \bar{\bar{\Phi}}_{r_1 r_1} \delta \bar{r}_1^2 + \bar{\bar{\Phi}}_{\theta_1 r_1} \delta \bar{\theta}_1 \delta \bar{r}_1$$

to which (13.16) is easily reduced if the substitution

$$(13.57)\quad \delta\bar{v}_1 = \frac{1}{\bar{\Phi}_{v_1}}\left(\delta^1\Phi - \bar{\Phi}_{\theta_1}\delta\bar{\theta}_1 - \bar{\Phi}_{r_1}\delta\bar{r}_1\right)$$

is made on the basis of (13.4). The following expressions are here obtained for the coefficients in front of the products of the deviations:

$$(13.58)\quad \bar{\bar{\Phi}}_{\Phi\Phi} = \frac{1}{2}\,\varphi_2 - \frac{3}{2}\,\frac{1}{\bar{\Phi}_{v_1}}$$

$$(13.59)\quad \bar{\bar{\Phi}}_{\Phi\theta_1} = 2\tan\theta_1 - \varphi_1(1 + \tan^2\theta_1) + 3\,\frac{\bar{\Phi}_{\theta_1}}{\bar{\Phi}_{v_1}}$$

$$= 3k_1\cot\frac{\Phi}{2} - \tan\theta_1 - \varphi_1(1 + \tan^2\theta_1)$$

$$(13.60)\quad \bar{\bar{\Phi}}_{\Phi r_1} = -\left[1 + \varphi_1(1 + \bar{h}_1)\cot\frac{\Phi}{2}\right] + 3\,\frac{\bar{\Phi}_{r_1}}{\bar{\Phi}_{v_1}}$$

$$= \frac{1}{2} + \varphi_1(1 + \bar{h}_1)\left(\frac{3}{\bar{\Phi}_{v_1}} - \cot\frac{\Phi}{2}\right)$$

$$(13.61)\quad \bar{\bar{\Phi}}_{\theta_1\theta_1} = -\frac{1}{2}(1 + \tan^2\theta_1)\bar{\Phi}_{v_1} - \tan\theta_1\,\bar{\Phi}_{\theta_1} - \frac{3}{2}\,\frac{\bar{\Phi}^2_{\theta_1}}{\bar{\Phi}_{v_1}}$$

$$= \left(\frac{1}{2}\tan\theta_1 - \frac{3}{2}k_1\cot\frac{\Phi}{2}\right)\bar{\Phi}_{\theta_1} - \frac{1}{2}(1 + \tan^2\theta_1)\,\bar{\Phi}_{v_1}$$

$$(13.62)\quad \bar{\bar{\Phi}}_{\theta_1 r_1} = \varphi_1\sin\Phi\,(1 + \tan^2\theta_1) - 2\tan\theta_1\,\bar{\Phi}_r - 3\,\frac{\bar{\Phi}_{\theta_1}}{\bar{\Phi}_{v_1}}\,\bar{\Phi}_{r_1}$$

$$= \varphi_1(1 + \tan^2\theta_1)\sin\Phi + \left(\tan\theta_1 - 3k_1\cot\frac{\Phi}{2}\right)\bar{\Phi}_{r_1}$$

$$(13.63)\quad \bar{\bar{\Phi}}_{r_1 r_1} = \varphi_1(1 + \bar{h}_1) - \frac{3}{2}\,\frac{\bar{\Phi}^2_{r_1}}{\bar{\Phi}_{v_1}}$$

# 14. DEVIATIONS IN FLIGHT TIME CAUSED BY INITIAL DEVIATIONS

In analyzing motions along Keplerian trajectories, we often use a dependence relating the deviations in the initial motion conditions to the flight-time deviations they cause. Applying the Taylor formula, as in the preceding case, to the function

$$(14.1)\quad T = T(v_1,\ \theta_1,\ r_1)$$

we can obtain this dependence in the following form:

$$(14.2)\quad \delta T = \sum_{k=1}^{n} \delta^k T + R_{n+1}$$

where

$$(14.3)\quad \delta^k T = \frac{1}{k!}\left(\delta v_1 \frac{\partial}{\partial v_1} + \delta\theta_1 \frac{\partial}{\partial \theta_1} + \delta r_1 \frac{\partial}{\partial r_1}\right)^k T$$

$R_{n+1}$ is the remainder term; $\delta v_1$, $\delta\theta_1$, $\delta r_1$ the initial deviations in the quantities $v$, $r$, $\theta$; $\delta T$ the flight-time deviation caused by the deviations $\delta v_1$, $\delta\theta_1$, $\delta r_1$.

Considering the initial deviations to be sufficiently small, it is possible to find an approximate expression of the desired dependence by neglecting terms higher than the first or second orders in Taylor's formula.

The expression for $\delta^1 T$ can be written as follows:

$$\delta^1 T = \bar{T}_{v_1} \delta\bar{v}_1 + \bar{T}_{\theta_1} \delta\bar{\theta}_1 + \bar{T}_{r_1} \delta\bar{r}_1 \tag{14.4}$$

where

$$\bar{T}_{v_1} = \frac{\partial T}{\partial v_1} v_1 \; ; \quad \bar{T}_{\theta_1} = \frac{\partial T}{\partial \theta_1} \; ; \quad T_{r_1} = \frac{\partial T}{\partial r_1} r_1 \tag{14.5}$$

$$\delta\bar{v}_1 = \frac{\delta v_1}{v_1} \; ; \quad \delta\bar{\theta}_1 = \delta\theta_1 \; ; \quad \delta\bar{r}_1 = \frac{\delta r_1}{r_1} \; . \tag{14.6}$$

In order to determine the quantities in (14.5) it is necessary to find the partial derivatives of T with respect to the parameters $r_1$, $\theta_1$, $v_1$. Let us determine these partial derivatives for the case of point motion along elliptical trajectories when the function (14.1) can be represented in the form of the relations

$$\left\{ \begin{aligned}
& T = \pm T_p \left[ \left( \frac{\pi}{2} - \beta_1 + \gamma_1 \right) \pm \left( \frac{\pi}{2} - \beta_2 + \gamma_2 \right) \right] \\
& T_p = \sqrt{\frac{a^2}{\pi_0}} \; ; \quad a = \frac{r_1}{2(1 - k_1)} \\
& \beta_1 = \arcsin \frac{1 - 2k_1}{e} \qquad \left( -\frac{\pi}{2} \leq \beta_1 \leq \frac{\pi}{2} \right) \\
& \beta_2 = \arcsin \frac{1 - 2k_2}{e} \qquad \left( -\frac{\pi}{2} \leq \beta_2 \leq \frac{\pi}{2} \right) \\
& \gamma_1 = e \cos \beta_1 \; ; \quad k_1 = \frac{v_1^2 r_1}{2\pi_0} \; ; \quad k_2 = \frac{k_1 + \hat{h}_1}{1 + \hat{h}_1} \\
& \gamma_2 = e \cos \beta_2 \; ; \quad \hat{h}_1 = \frac{r_1 - r_2}{r_2} \; ; \quad e^2 = (1 - 2k_1)^2 \cos^2\theta_1 + \sin^2\theta_1
\end{aligned} \right. \tag{14.7}$$

presented in section 9 in formulas (9.59) - (9.61).

For our purposes it is convenient to represent the first of the above relations in the form:

$$(14.7a)\quad T = (-1)^m T_p \left[\tau_1 + (-1)^n \tau_2\right]$$

where

$$(14.8)\quad \tau_1 = \frac{\pi}{2} + \gamma_1 - \beta_1 ; \quad \tau_2 = \frac{\pi}{2} + \gamma_2 - \beta_2 .$$

Values of the numbers m and n must be selected as follows:

1) If the points $O_1$ and $O_2$ between which the flight time is measured lie on different sides of the peak of the trajectory $O_F$, then it should be assumed that

$$(14.9)\quad m = 0 ; \quad n = 0 .$$

2) If the points $O_1$ and $O_2$ lie on the same side of the peak of the trajectory and $r_2 < r_1$, then it should be assumed that

$$(14.10)\quad m = 1 ; \quad n = 1 .$$

3) If the points $O_1$ and $O_2$ lie on the same side of the peak of the trajectory and $r_2 > r_1$, then it should be assumed that

$$(14.11)\quad m = 0 ; \quad n = 1 .$$

We find the derivatives of interest to us by differentiating (14.7). Hence, as in the preceding cases, let us determine the partial derivatives with respect to all the parameters. Denoting the variable with respect to which the differentiation is to be carried out by q, we can write

$$(14.12)\quad \frac{\partial T}{\partial q} = \frac{1}{T_p}\frac{\partial T_p}{\partial q} T + (-1)^m T_p \left[\frac{\partial \tau_1}{\partial q} + (-1)^n \frac{\partial \tau_2}{\partial q}\right] .$$

It is easy to note that

$$(14.13)\quad \frac{\partial T_p}{\partial q} = T_p \left[\frac{3}{2a}\frac{\partial a}{\partial q} - \frac{1}{2\pi_0}\frac{\partial \pi_0}{\partial q}\right]$$

Before determining the derivative $\partial a/\partial q$ it is convenient to write the derivatives of $k_1$ and $k_2$:

$$(14.14)\quad \frac{\partial k_1}{\partial q} = k_1\left(\frac{2}{v_1}\frac{\partial v_1}{\partial q} + \frac{1}{r_1}\frac{\partial r_1}{\partial q} - \frac{1}{\pi_0}\frac{\partial \pi_0}{\partial q}\right)$$

$$(14.15)\quad \frac{\partial k_2}{\partial q} = \frac{1}{1 + \bar{h}_1}\left(\frac{2k_1}{v_1}\frac{\partial v_1}{\partial q} + \frac{1}{r_1}\frac{\partial r_1}{\partial q} - \frac{k_1}{\pi_0}\frac{\partial \pi_0}{\partial q} + \frac{k_1 - 1}{r_2}\frac{\partial r_2}{\partial q}\right) .$$

Differentiating the expression for a, we obtain

$$(14.16)\quad \frac{\partial a}{\partial q} = a\left(\frac{1}{r_1}\frac{\partial r_1}{\partial q} + \frac{1}{1 - k_1}\frac{\partial k_1}{\partial q}\right)$$

Substituting (14.14) into (14.16), we can rewrite it

$$(14.17)\quad \frac{\partial a}{\partial q} = a\left(\frac{1}{r_1(1 - k_1)}\frac{\partial r_1}{\partial q} + \frac{2k_1}{v_1(1 - k_1)}\frac{\partial v_1}{\partial q} - \frac{k_1}{\pi_0(1 - k_1)}\frac{\partial \pi_0}{\partial q}\right)$$

Using (14.17), we can represent (14.13) as follows:

$$(14.18)\quad \frac{\partial T_p}{\partial q} = T_p\left[\frac{3}{2r_1(1 - k_1)}\frac{\partial r_1}{\partial q} + \frac{3k_1}{v_1(1 - k_1)}\frac{\partial v_1}{\partial q} - \frac{1 + 2k_1}{2\pi_0(1 - k_1)}\frac{\partial \pi_0}{\partial q}\right]$$

from which

$$(14.19)\quad \begin{cases} \dfrac{\partial T_p}{\partial v_1} = \dfrac{3}{v_1}\dfrac{k_1}{1 - k_1}T_p \; ; \quad \dfrac{\partial T_p}{\partial \theta_1} = 0 \; ; \quad \dfrac{\partial T_p}{\partial \pi_0} = - \dfrac{1 + 2k_1}{2\pi_0(1 - k_1)}T_p \\[2ex] \dfrac{\partial T_p}{\partial r_1} = \dfrac{3}{2r_1(1 - k_1)}T_p \, , \quad \dfrac{\partial T_p}{\partial r_2} = 0 \end{cases}$$

Differentiating the expressions for $\tau_1$ and $\gamma_1$, we obtain:

$$(14.20)\quad \frac{\partial \tau_1}{\partial q} = \frac{\partial \gamma_1}{\partial q} - \frac{\partial \beta_1}{\partial q}$$

(14.21) $$\frac{\partial \gamma_1}{\partial q} = \frac{\partial e}{\partial q} \cos \beta_1 - e \sin \beta_1 \frac{\partial \beta_1}{\partial q} ,$$

after which, substituting (14.21) into (14.20), we can write:

(14.22) $$\frac{\partial \tau_1}{\partial q} = \frac{\partial e}{\partial q} \cos \beta_1 - (1 + e \sin \beta_1) \frac{\partial \beta_1}{\partial q}$$

Differentiating the equality

(14.23) $$\sin \beta_1 = \frac{1 - 2k_1}{e}$$

we can write

(14.24) $$- \frac{\partial \beta_1}{\partial q} = \frac{\sin \beta_1}{e \cos \beta_1} \frac{\partial e}{\partial q} + \frac{2}{e \cos \beta_1} \frac{\partial k_1}{\partial q} .$$

Substituting (14.24) into (14.22), we arrive at the expression

(14.25) $$\frac{\partial \tau_1}{\partial q} = \frac{1}{e \cos \beta_1} \left[ (\sin \beta_1 + e) \frac{\partial e}{\partial q} + 2(1 + e \sin \beta_1) \frac{\partial k_1}{\partial q} \right] ,$$

or taking into account that

(14.26) $$\gamma_1 = e \cos \beta_1 ; \quad 1 + e \sin \beta_1 = 2(1 - k_1) ,$$

at the following expression:

(14.27) $$\frac{\partial \tau_1}{\partial q} = \frac{1}{\gamma_1} \left[ (\sin \beta_1 + e) \frac{\partial e}{\partial q} + 4(1 - k_1) \frac{\partial k_1}{\partial q} \right] .$$

We obtain a similar expression for the derivative of $\tau_2$ analogously:

(14.28) $$\frac{\partial \tau_2}{\partial q} = \frac{1}{\gamma_2} \left[ (\sin \beta_2 + e) \frac{\partial e}{\partial q} + 4(1 - k_2) \frac{\partial k_2}{\partial q} \right] .$$

Differentiating the expression

$$(14.29)\quad e^2 = (1 - 2k_1)^2\cos^2\theta_1 + \sin^2\theta_1$$

let us write

$$(14.30)\quad \frac{\partial e}{\partial q} = \sin\theta_1 \cos\theta_1 \frac{4k_1(1 - k_1)}{e} \frac{\partial\theta_1}{\partial q} - 2\cos^2\theta_1 \frac{1 - 2k_1}{e} \frac{\partial k_1}{\partial q}\,.$$

Let us use the notation

$$(14.31)\quad \tau = \tau_1 + (-1)^n\tau_2\ ;\ b_1 = \sin\beta_1 = \frac{1 - 2k_1}{e}\ ;$$

$$b_2 = \sin\beta_2 = \frac{1 - 2k_2}{e}$$

then

$$(14.32)\quad \frac{\partial\tau}{\partial q} = \frac{\partial\tau_1}{\partial q} + (-1)^n \frac{\partial\tau_2}{\partial q} = \frac{\partial e}{\partial q}\left[\frac{b_1 + e}{\gamma_1} + (-1)^n \frac{b_2 + e}{\gamma_2}\right]$$

$$+ 4\left[\frac{1 - k_1}{\gamma_1} \frac{\partial k_1}{\partial q} + (-1)^n \frac{1 - k_2}{\gamma_2} \frac{\partial k_2}{\partial q}\right]$$

or

$$(14.33)\quad \frac{\partial\tau}{\partial q} = 4(1 - k_1)\left[\frac{1}{\gamma_1} \frac{\partial k_1}{\partial q} + \frac{(-1)^n}{1 + \bar{h}_1} \frac{1}{\gamma_2} \frac{\partial k_2}{\partial q}\right]$$

$$+ \frac{\partial e}{\partial q}\left[\frac{b_1 + e}{\gamma_1} + (-1)^n \frac{b_2 + e}{\gamma_2}\right]$$

since

$$(14.34)\quad 1 - k_2 = \frac{1 - k_1}{1 + \bar{h}_1}$$

Using (14.33), (14.30), (14.14) and (14.15), we can write

$$
(14.35)\left\{
\begin{aligned}
&\frac{\partial\tau}{\partial v_1} = \frac{2}{v_1}(k_1\gamma_3 - b_3)\;; \quad \frac{\partial\tau}{\partial\theta_1} = \tan\theta_1\left(1 + \frac{1}{b_1 e}\right)b_3 \\
&\gamma_3 = 4(1 - k_1)\left[\frac{1}{\gamma_1} + \frac{(-1)^n}{(1 + \bar{h}_1)^2}\frac{1}{\gamma_2}\right] \\
&\frac{\partial\tau}{\partial r_1} = \frac{1}{r_1}(\gamma_4 - b_3)\;; \quad \gamma_4 = 4(1 - k_1)\left[\frac{k_1}{\gamma_1} + \frac{(-1)^n}{(1 + \bar{h}_1)^2}\frac{1}{\gamma_2}\right] \\
&\frac{\partial\tau}{\partial\pi_0} = \frac{1}{\pi_0}(-k_1\gamma_3 + b_3)\;; \quad b_3 = 2\cos^2\theta_1\, b_1 k_1\left[\frac{b_1 + e}{\gamma_1}\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. + (-1)^n\frac{b_2 + e}{\gamma_2}\right] \\
&\frac{\partial\tau}{\partial r_2} = \frac{(-1)^{n+1}}{r_2}\frac{(1 - k_1)^2}{(1 + \bar{h}_1)^2}\frac{1}{\gamma_2}\,.
\end{aligned}
\right.
$$

Then substituting (14.35) and (14.19) into (14.12), we obtain the desired derivatives in the following form:

$$
(14.36)\left\{
\begin{aligned}
&\frac{\partial T}{\partial v_1} = \left[3\frac{k_1}{1 - k_1}T + 2(-1)^m T_p(k_1\gamma_3 - b_3)\right]\frac{1}{v_1} \\
&\frac{\partial T}{\partial\theta_1} = (-1)^m T_p \tan\theta_1\left(1 + \frac{1}{b_1 e}\right)b_3 \\
&\frac{\partial T}{\partial r_1} = \left[\frac{3}{2(1 - k_1)}T + (-1)^m T_p(\gamma_4 - b_3)\right]\frac{1}{r_1} \\
&\frac{\partial T}{\partial\pi_0} = \left[-\frac{1 + 2k_1}{2(1 - k_1)}T + (-1)^m T_p(-k_1\gamma_3 + b_3)\right]\frac{1}{\pi_0} \\
&\frac{\partial T}{\partial r_2} = (-1)^{m+n+1}\frac{(1 - k_1)^2}{(1 + \bar{h}_1)^2}\frac{1}{\gamma_2}\frac{1}{r_2}
\end{aligned}
\right.
$$

where

$$(14.37)\left\{\begin{aligned}
&k_1 = \frac{v_1^2 r_1}{2\pi_0}\ ;\quad \gamma_1 = e\sqrt{1 - b_1^2}\ ;\quad \gamma_2 = e\sqrt{1 - b_2^2}\\
&b_1 = \frac{1 - 2k_1}{e}\ ;\quad b_2 = \frac{1 - 2k_2}{e}\ ;\quad k_2 = \frac{k_1 + \bar{h}_1}{1 + \bar{h}_1}\\
&\bar{h}_1 = \frac{r_1 - r_2}{r_2}\ ;\quad e = \sqrt{1 - 4k_1(1 - k_1)\cos^2\theta_1}\\
&a = \frac{r_1}{2(1 - k_1)}\ ;\quad T_p = a\sqrt{\frac{a}{\pi_0}}\\
&\beta_1 = \arcsin b_1\ ;\ \beta_2 = \arcsin b_2\ \left(-\frac{\pi}{2} \le \beta_1 \le \frac{\pi}{2};\ -\frac{\pi}{2} \le \beta_2 \le \frac{\pi}{2}\right)\\
&T = (-1)^m T_p\left[\tau_1 + (-1)^n \tau_2\right]\ ;\ \tau_1 = \frac{\pi}{2} + \gamma_1 - \beta_1\ ;\\
&\tau_2 = \frac{\pi}{2} + \gamma_2 - \beta_2\\
&\gamma_3 = 4(1 - k_1)\left[\frac{1}{\gamma_1} + \frac{(-1)^n}{(1 + \bar{h}_1)^2}\,\frac{1}{\gamma_2}\right];\\
&\gamma_4 = 4(1 - k_1)\left[\frac{k_1}{\gamma_1} + \frac{(-1)^n}{(1 + \bar{h}_1)^2}\,\frac{1}{\gamma_2}\right]\\
&b_3 = 2b_1k_1\cos^2\theta_1\left[\frac{b_1 + e}{\gamma_1} + (-1)^n\,\frac{b_2 + e}{\gamma_2}\right]
\end{aligned}\right.$$

As already mentioned, in certain cases the deviations in flight time caused by initial deviations are conveniently represented thus:

$$(14.38)\quad \delta^1 T = \bar{\bar{T}}_{\Phi}\delta^1\Phi + \bar{\bar{T}}_{\theta_1}\delta\bar{\theta}_1 + \bar{\bar{T}}_{r_1}\delta\bar{r}_1$$

to which (14.4) is easily reduced if the substitution (13.57) is made therein. The following expressions are hence obtained for the coefficients $\bar{\bar{T}}_{\Phi}$, $\bar{\bar{T}}_{\theta_1}$, $\bar{\bar{T}}_{r_1}$:

$$(14.39)\quad \bar{\bar{T}}_{\Phi} = \frac{\bar{T}_{v_1}}{\bar{\Phi}_{v_1}}\,;\ \bar{\bar{T}}_{r_1} = \bar{T}_{r_1} - \bar{T}_{v_1}\frac{\bar{\Phi}_{r_1}}{\bar{\Phi}_{v_1}}\,;\ \bar{\bar{T}}_{\theta_1} = \bar{T}_{\theta_1} - \bar{T}_{v_1}\frac{\bar{\Phi}_{\theta_1}}{\bar{\Phi}_{v_1}}$$

where

$$(14.40)\quad \begin{cases} \bar{T}_{v_1} = \left[\dfrac{3k_1}{1 - k_1}\,T + 2(-1)^m T_p(k_1\gamma_3 - b_3)\right] \\[2ex] \bar{T}_{\theta_1} = (-1)^m T_p \tan\theta_1\left(1 + \dfrac{1}{b_1 e}\right)b_3 \\[2ex] \bar{T}_{r_1} = \left[\dfrac{3}{2(1 - k_1)}\,T + (-1)^m T_p(\gamma_4 - b_3)\right] \end{cases}$$

and the rest of the quantities in (14.39) and (14.40) are defined by formulas (13.14) and (14.37).

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